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SCHEDULING RESERVOIR OIL PRODUCTION BY LINEAR PROGRAMMING AND THE NEYMAN-PEARSON LEMMA

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Scheduling Reservoir Oil Production by Linear Programming and the Neyman-Pearson Lemma

1. STATEMENT OF THE PROBLEM

In a set of n oil wells located in reservoirs without communications, we wish to schedule the flowrate of each individual well in such a way that we satisfy the following conditions:

-- the pressure drop of each individual well is less than or equal to a given value,

-- the production of any well is bounded by lower and upper technical limits,

-- the total production of the fields has to satisfy a certain demand with some specific technical limiting capacities,

-- the total discounted profit of the oil production is maximized.

Following the basic fluid and cash flow mechanisms of the system, two flowrate arrangements will be considered, namely:

-- discrete flowrate,

-- continuous flowrate.

They will be applied to systems including single or multiwells through a linear programming model and an attempt to use the Neyman-Pearson lemma.

1.1 FLUID FLOW

Besides limitations on the production of each part of a given oil field, a special consideration has to be made for the maximum pressure drop in the reservoir. Since the maximum is located at the bottom of the well bore, this region will have to satisfy at any time the pressure constraints.

The concept of a well function defined as the bottom hole pressure drop at time t_x per unit of constant flowrate is written as

$$F_j(t_x) \quad \text{where} \quad F_j(0) = 0 \qquad 1.101$$

and due to the linearity of the diffusivity equation, we can use the superposition principle or convolution method as:

$$P_{0,j} - P_{x,j} = \sum_{i=1}^{n} (q_{i,j} - q_{i-1,j})F_j(t_x - t_{i,j}) \qquad 1.102$$

We want to satisfy for the well j,

$$P_{0,j} - P_{x,j} \leq \delta_j \qquad 1.103$$

therefore, in the case of a discrete flowrate,

$$\boxed{\sum_{i=1}^{n}(q_{i,j} - q_{i-1,j})F_j(t_x - t_{i,j}) \leq \delta_j} \qquad 1.104$$

and without loss of generality, we can assume

$$t_{i,j} - t_{i-1,j} = \Delta t = \frac{T}{n}$$

and

$$\sum_{i=1}^{n} \frac{q_{i,j} - q_{i-1,j}}{\Delta t} \Delta t F_j(t_x - t_{i,j}) \leq \delta_j \qquad 1.105$$

or, at the limit as $\Delta t \rightarrow 0$ for a continuous flowrate,

$$\boxed{\int_{t=0}^{t=t_x} q_{i,j}'(t)F_j(t_x - t_{i,j})dt \leq \delta_j} \qquad 1.106$$

where

$$q_{i,j}'(t) = \lim_{\substack{n \to \infty \\ \text{or } \Delta t \to 0}} \left(\frac{q_{i,j} - q_{i-1,j}}{\Delta t} \right)$$

The three well functions which will be used in the calculation are represented in figs. 1 and 2.

Maximum and minimum limitations

Uniquely referring to the reservoir capacities with some pressure drop imposed, we can define two limitations on the flowrate, namely:

max Q_{max}: constant flowrate during any time period which will produce the maximum permissible drop at the end of the period.

max Q_{min}: constant flowrate during the entire time which will produce the maximum permissible pressure drop at the end of the entire period.

For a well j:

$$\max q_{max} = \frac{\delta_j}{F_j(\frac{T}{n})} \quad \text{and} \quad \max q_{min} = \frac{\delta_j}{F_j(T)} \qquad 1.107$$

For a multiwell system which includes m wells:

$$\max Q_{max} = \sum_{j=1}^{m} \frac{\delta_j}{F_j(\frac{T}{n})} \quad \text{and} \quad \max Q_{min} = \sum_{j=1}^{m} \frac{\delta_j}{F_j(T)} \qquad 1.108$$

The technical capacities and the demands must be such that:

min (max q_{min}, min technical capacities well j) $\leq$ individual flowrate $q_{x,j}$ 1.109

min (max q_{max}, max technical capacities well j) $\geq$ individual flowrate $q_{x,j}$ 1.110

min (max Q_{min}, min total technical capacities) $\leq$ total flowrate Q_x 1.111

min (max Q_{max}, max total technical capacities $\geq$ total flowrate Q_x 1.112

and

$$\text{demand} \leq \text{total flowrate} \qquad 1.113$$

1.2 CASH FLOW

Between times t_i and t_{i+1} when the flowrate is $q_{i,j}$ the profit is such that:

$$\text{Profit} = \int_{t_i}^{t_{i+1}} e^{-\lambda T\tau} \, q_{i,j}(t)dt \qquad 1.201$$

where the discount function is $e^{-\lambda T\tau}$ and where $\tau = \frac{t_x}{T}$

Let $\alpha = \lambda T$ and $\gamma = e^{-\alpha}$

or $\tau = 0 \Rightarrow e^{-\lambda T\tau} = 1$

and $\tau = 1 \Rightarrow e^{-\lambda T\tau} = \gamma$

Four cases will be considered here:

$$\gamma_0 = 1.00 \Rightarrow \alpha_0 = 0.000$$
$$\gamma_1 = 0.90 \Rightarrow \alpha_1 = 0.105$$
$$\gamma_2 = 0.75 \Rightarrow \alpha_2 = 0.288$$
$$\gamma_3 = 0.50 \Rightarrow \alpha_3 = 0.693$$

The corresponding discount functions are shown on fig. 3.

Total discounted profit

a) Discrete flowrate: assuming a constant flowrate $q_{i,j}(t)$ between t_i and t_{i+1}:

$$\text{Profit} = \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} e^{-\frac{\alpha t}{T}} \, q_{i,j}(t)dt \qquad 1.202$$

$$\text{Total discounted profit} = \sum_{\substack{i=1 \\ n\varepsilon T}}^{n} q_{i,j}(t)\frac{T}{\alpha}\left(e^{-\alpha\tau_i} - e^{-\alpha\tau_{i+1}}\right) \qquad 1.203$$

Let $T = n\Delta t$ and $\tau_i{}^* = \dfrac{\tau_i + \tau_{i+1}}{2}$

then

$$\boxed{\text{Total discounted profit} = \frac{T}{n}\sum_{i=1}^{n} e^{-\alpha\tau_i{}^*} q_{i,j}(t)} \qquad 1.204$$

b) Continuous flowrate:

$$\boxed{\text{Total discounted profit} = \int_0^T e^{-\frac{\alpha t}{T}}\, q_{i,j}(t)dt} \qquad 1.205$$

The profit function has the dimension of dollars and it represents the total profit over the entire lifetime of the project.

2. INDIVIDUAL WELL PROBLEM

We consider one well in a reservoir and it is required that the oil production of the given well be scheduled during a time T. In a first linear program model, the total time T is divided into a series of n time periods Δt and $q_{i,j}(t)$ is computed. Then, the Neyman-Pearson lemma will be applied in the case of a continuous flowrate for the determination of the exact function $q_j(t)$.

It is obvious that, in both cases, the cumulative production, namely,

$$\sum_{\substack{i=1 \\ n\varepsilon T}}^{n} q_{i,j}(t)\Delta t_i$$

or

$$\int_0^T q_j(t)dt$$

cannot be greater than the reserves of the reservoir. However, this

last constraint will not be considered here, and we assume that for any well j we satisfy:

$$\sum_{i=1}^{n} q_{i,j}(t)\Delta t_i \leq R_j \quad \text{or} \quad \int_0^T q_j(t)dt \leq R_j \tag{2.001}$$

where R_j are the reserves for the well j.

2.1 DISCRETE FLOWRATE

From the preceding statements, in the case of a discrete flowrate the function $q_{i,j}(t)$ is the solution of the following system:

1) maximize $\frac{T}{n} \sum_{i=1}^{n} e^{-\alpha\tau_i^*} q_{i,j}(t)$ 2.101

over all functions $q_{i,j}(t)$ subject to

2) the pressure constraints:

$$\sum_{i=1}^{n} q_{i,j}(t)\left\{F_j\left[(N - i+1)\Delta t\right] - F_j\left[(N - i)\Delta t\right]\right\} \leq \delta_j \tag{2.102}$$

$$N = 1, \ldots, n$$

or

$$\sum_{i=1}^{n} q_{i,j}(t)\left\{\bar{\omega}_j\left[(N - i+1)\Delta t\right] - \bar{\omega}_j\left[(N - i)\Delta t\right]\right\} \leq \Delta_j \tag{2.103}$$

where $\Delta_j = \frac{2\pi hk}{\eta} \delta_j$

3) bounded by some technical limiting capacities as:

$$q_{i,j}\min(t) \leq q_{i,j}(t) \leq q_{i,j}\max(t) \tag{2.104}$$

4) where the entire lifetime of the project is defined as:

$$0 \leq t_i \leq T \qquad 2.105$$

This problem is easily solved by the use of computers and well-improved linear programming techniques. The solutions obtained, however, show a dependence under the selected value of n. The true maximum could be obtained by greatly increasing n. But the size of our problem is fast becoming large and lengthy. It is therefore interesting to try to determine the actual true maximum by another method dealing with continuous flowrates.

2.2 CONTINUOUS FLOWRATE

In the case of a continuous flowrate, the function $q_j(t)$ is the solution of the following system:

1) maximize

$$\int_0^T e^{-\frac{\alpha t}{T}} q_j(t)dt \qquad 2.201$$

over all functions $q_j(t)$ subject to

2) the pressure constraint:

$$\int_0^t q_j'(t)\bar{\omega}_j(t_x - t)dt \leq \Delta_j \qquad 2.202$$

3) bounded by some technical limiting capacities, as

$$q_j\min \leq q_j(t) \leq q_j\max \qquad 2.203$$

4) where the entire lifetime of the project is defined as

$$0 \leq t \leq T \qquad 2.204$$

The above system will be somewhat modified through a change of variables, in order to take advantage of the Neyman-Pearson lemma as presented in appendix 5.1 and by fig. 14.

Let $$\tau = \frac{t}{T}$$

then $$0 \leq \tau \leq 1 \qquad 2.205$$

and $$q_j(\tau) = \frac{q_j(\tau T)}{q_j max} \qquad 2.206$$

or $$0 \leq q_j(\tau) \leq 1 \qquad 2.207$$

Now consider the equation 2.201 which is equivalent to

$$\text{minimize} \int_0^T - e^{\alpha \frac{t}{T}} q_j(t)dt$$

Or, if we let

$$a(\tau) = -e^{-\alpha\tau} \qquad 2.208$$

the objective function of the system is

$$\text{minimize} \int_0^1 a(\tau)q_j(\tau)d\tau \qquad 2.209$$

Consider the function I(t) defined as

$$I(t) = \int_0^t q_j'(t)\bar{\omega}_j(T - t)dt$$

and let us transform this function in the following way:

$$I_j(\tau) = \int_0^\tau q_j'(T\tau)\bar{\omega}_j\Big[T(1 - \tau)\Big]Td\tau$$

$$\frac{dq_j(\tau)}{d\tau} = \frac{1}{q_j max} q_j'(T\tau)t_x \Rightarrow q_j'(T\tau) = \frac{dq_j(\tau)}{d\tau} \cdot \frac{q_j max}{T}$$

and, finally:

$$I_1(\tau) = q_j\max \int_0^{\tau} \frac{dq_j(\tau)}{d\tau} \bar{\omega}_j\left[T(1 - \tau)\right]d\tau \qquad 2.210$$

Integrating by parts, equation 2.202 can be stated as

$$\left\{q_j(\tau)\bar{\omega}_j\left[T(1 - \tau)\right]\right\}\Big|_0^{\tau} - \int_0^{\tau} q(\tau)\frac{d\bar{\omega}_j\left[T(1 - \tau)\right]}{d\tau}\,d\tau \leq \frac{\Delta_j}{q_j\max} \qquad 2.211$$

Let

$$b(\tau) = -\frac{d\bar{\omega}_j\left[T(1 - \tau)\right]}{d\tau} \qquad 2.212$$

and

$$c(\tau) = \frac{\Delta_j}{q_j\max} - \left\{q_j(\tau)\bar{\omega}_j\left[T(1 - \tau)\right]\right\}\Big|_0^{\tau} \qquad 2.213$$

or

$$c(\tau) = \frac{\Delta_j}{q_j\max} + q_j(0)\bar{\omega}_j(T) - q_j(\tau)\bar{\omega}_j\left[T(1 - \tau)\right] \qquad 2.214$$

Our initial system is now the following:

$$\text{minimize} \int_0^1 a(\tau)q_j(\tau)d\tau \qquad 2.215$$

over all functions $q_j(t)$ subject to the constraints:

a) $0 \leq q_j(\tau) \leq 1, \; 0 \leq \tau \leq 1$ 2.216

b) $\int_0^{\tau} b(\tau)q_j(\tau)d\tau \leq c(\tau)$ 2.217

where $a(\tau)$, $b(\tau)$, and $c(\tau)$ are given functions such as

$$a(\tau) = -e^{-\alpha\tau} \qquad 2.218$$

$$b(\tau) = -\frac{d\bar{\omega}_j\left[T(1-\tau)\right]}{d\tau} \qquad 2.219$$

$$c(\tau) = \frac{\Delta_j}{q_j\max} + q_j(0)\omega_j(T) - q_j(\tau)\omega_j\left[T(1-\tau)\right] \qquad 2.220$$

This form of the system allows free use of the Neyman-Pearson lemma, from which we obtain the optimum solution.

2.3 PROPOSED SOLUTIONS

A numerical solution is now presented for the case of an individual well surrounded by a radial circular reservoir. The derivation of the well function of such a system is shown in appendix 5.2. Three different wells will be considered. In this first part only one single well, say well j, is included in the system that we want to schedule.

All the units in this study are expressed in terms of the centimeter-gram-second system, the results being shown in practical units.

Characteristics of the wells

1) thickness of the formation: $h = 10^3$ cm
2) well radius: $a = 10$ cm
3) outside reservoir radius: $R = 2 \times 10^5$ cm or $R \geqq 1$ mile
4) porosity: $\omega = 10\%$
5) permeability:

well (1): $k = 10^{-12}$ CGS or 0.1 millidarcy

well (2): $k = 10^{-11}$ CGS or 1.0 millidarcy

well (3): $k = 10^{-10}$ CGS or 10.0 millidarcy

6) compressibility of the fluids: $\beta = 10^{-10}$ CGS (cm^2 per dyne)

7) viscosity of the fluids: $\mu = 10^{-2}$ CGS (Poise or dyne-second per cm^2)

The maximum pressure drop allowed here is of 10^8 dynes or 1,450 psi. The numerical solution presented covers a time interval of 1 year. As can be seen, the flowrate profiles are independent of the economic conditions imposed in the system.

2.31 DISCRETE FLOWRATE LINEAR PROGRAMMING MODEL

We want to solve equations 2.101, 2.102, and 2.103 for a well j, where

$$T = 1 \text{ year} \quad \text{and} \quad n = 12$$

Well function

In this case the function $\bar{\omega}_3(\theta)$ (see appendix 5.2, equation 5.211) can be used:

$$\bar{\omega}_3(\theta) = \frac{1}{2}(\text{Log } \theta + 0.80907)$$

Or let

$$F_j(t) = \bar{a} \log (t) + \bar{b} \qquad 2.3101$$

where:

	$\bar{a}$	$\bar{b}$
well (1)	1.83239×10^6	-1.18852×10^6
well (2)	1.83239×10^5	0.6438×10^5
well (3)	1.83239×10^4	2.4762×10^4

Maximum and minimum flowrates

From equation 1.107 and when $\delta_j = 10^8$ dynes:

	$F_j(T)$	$F_j(T/6)$	$F_j(T/12)$	Maximum (CGS)	Max q_{max}		
					n = 1	n = 6	n = 12
well (1)	12.541×10^6	11.115×10^6	10.564×10^6	7.974	7.974	8.997	9.466
well (2)	14.373×10^5	12.947×10^5	12.396×10^5	69.575	69.575	77.238	80.671
well (3)	16.206×10^4	14.780×10^4	14.229×10^4	617.055	617.055	676.589	702.790

Assuming that the technical capacities are such that

minimum technical capacities = 0.0

and

maximum technical capacities > max $q_j$max

equations 1.109 and 1.110 are reduced to

$$0 \leq \text{flowrate } q_j \leq \max q_j\max \qquad 2.3102$$

Solution

The solution of this simple linear programming problem is easily obtained by using the Share program number HOSCM3 on the 7094 IBM computer.

In fig. 4 a case is presented where upper and lower limits exist on the individual flowrates for well (2), assuming four different discount functions, namely, α_0, α_1, α_2, α_3, as defined in section 1.2.

The optimum flowrate expressed as the ratio q average/q max for different values of q max max/q_{max} is shown on fig. 5 for the three wells. The profit is also represented, and the influence of the number n on the final results can be seen.

2.32 CONTINUOUS FLOWRATE NEYMAN-PEARSON LEMMA

From equations 5.201 and 2.219,

$$b(\tau) = 2\sum_{n=1}^{4} \alpha_n \beta_n^{2}\left(\frac{a^2}{K}\right) e^{-\frac{\beta_n^2 a^2}{K}\left[T(1-\tau)\right]} \qquad 2.3201$$

from equation 2.220,

$$c(\tau) = \frac{\Delta_j}{q_j\max} + q_j(0)\left[\text{Log } R' - 2\sum \alpha_n e^{-\frac{\beta_n^2 a^2}{K} T}\right]$$

$$- q_j(\tau)\left[\text{Log } R' - 2\sum \alpha_n e^{-\frac{\beta_n^2 a^2}{K} T(1-\tau)}\right] \qquad 2.3202$$

and from equation 5.1104,

$$A(\tau) = 2\left[\sum \alpha_n e^{-\frac{\beta_n^2 a^2}{K} T(1-\tau)} - \sum \alpha_n e^{-\frac{\beta_n^2 a^2}{K} T}\right] \qquad 2.3203$$

These three functions, $b(\tau)$, $c(\tau)$, and $A(\tau)$ are well defined for a given well and present some interesting properties. For instance, $b(\tau)$ being a monotonically non-negative increasing function with a finite value when $\tau = 0$, and $a(\tau)$ having also a finite value when $\tau = 0$, the supremum value of α_0', as defined in appendix 5.1, is such that

$$\alpha_0' = -\frac{e^{-\alpha}}{b(1)} \qquad 2.3204$$

Therefore, τ_1, the intersection of $a(\tau)$ and $\alpha_0'b(\tau)$, is equal to one. Thus a given well will be flowing at its maximum flowrate from $\tau = 0$ to $\tau = \tau_2$, τ_2 being defined as the intersection of $A(\tau)$ and $c(\tau)$. Then it is only necessary to determine τ_2 and to verify that $A(\tau) \geq c(\tau)$ when $0 \leq \tau \leq \tau_2$.

If we notice also that $A(0) = 0$ and $c(0) = \dfrac{\Delta_j}{q_j\text{max}} > 0$ then

$$q_j(0) = 1 \qquad 2.3205$$

This means that the well has to be set at its maximum value at zero time. This maximum value will remain up to the time τ_2, and the intersection of $A(\tau)$ and $c(\tau)$ can be determined by setting $q_j(\tau) = 1$ in equation 2.3202. However, a careful observation of the problem shows that we have to satisfy equation 2.217, where one of the limits is a function of τ. Therefore, in some cases, it could be hazardous to apply the Neyman-Pearson lemma systematically. A more complete method of optimal control, as the maximum principle of Pontriagine, would be preferable (ref. 5,6). This limitation restricts, to a certain extent, a simple application of the Neyman-Pearson lemma for this particular problem.

3 MULTIWELL SYSTEM

The multiwell system is considered here as an extension of the individual case when m wells are producing from a given set of reservoirs. When more than one well acts on the same system, however, interference effects are frequently observed. These effects will not be taken into account here; only the simplified case where all the wells are independent will be developed, assuming that each well has the same effect as if it were alone in the system.

Therefore, the cumulative production will consist of the linear summation of all the wells and the total cash flow will mediate the corresponding production with its economics.

3.1 DISCRETE FLOWRATE

Initially, it was thought that a generalization of the fundamental lemma of Neyman and Pearson might be used for the treatment of the multiwell system (ref. 4). However, due to the above-mentioned limitations regarding the individual well, this method had to be used with caution. It would be more advisable to use the optimum control principles presented in references 5 or 6.

Thus, this study will be concerned only with some aspects of the discrete case where the problem to be solved is the following:

-- Find the optimum flowrate of a multiwell system including m wells, with the following constraints:

-- pressure drop maximum at every well

-- upper and lower bounds on the wells

-- a certain demand to be satisfied

Then the formulation of the problem can be set as:

1) maximize

$$\sum_{j=1}^{m} \frac{T}{n} \sum_{i=1}^{n} e^{-\alpha \tau_i^*} q_{i,j}(t) \tag{3.1}$$

over all functions $q_{i,j}(t)$ subject to

$$\text{2)} \quad \sum_{j=1}^{m} \sum_{i=1}^{n} q_{i,j}(t) \left\{ \bar{\omega}_j \left[(N - i + 1)\Delta t \right] - \bar{\omega}_j \left[(N - i)\Delta t \right] \right\} \leq \Delta_j$$

$$N = 1, \ldots, n \tag{3.2}$$

3) bounded by some technical limiting capacities as

$$q_{i,j}\min(t) \leq q_{i,j}(t) \leq q_{i,j}\max(t) \tag{3.3}$$

4) $$\sum_{j=1}^{m} q_{i,j}(t) \leq Q(t) \qquad 3.4$$

5) where the entire lifetime of the project is defined as

$$0 \leq t \leq T \qquad 3.5$$

3.2 PROPOSED SOLUTION: LINEAR PROGRAMMING MODEL

This linear programming model has been solved for different cases. The basic matrix is presented in fig. 6, where:

$$n = 6, \quad m = 3, \quad q_{i,j}\text{min} = 0$$

$$q_{i,j}\text{max} = \infty, \quad Q(t) = \text{constant}, \quad \Delta_j = 10^8 \text{ dynes}$$

Due to the number of constraints, the size of the matrix increases as n is increased. So, when

n = 1 we get a matrix of 8 rows × 16 columns

n = 6 we get a matrix of 28 rows × 46 columns

n = 12 we get a matrix of 52 rows × 82 columns

and when n is large, the problem is greatly simplified by using the decomposition principle for linear programming.

Also, the number of rows and columns in the basic matrix determines whether the primal or the dual type of problem should be used, the fewer the number of rows in the matrix, the easier to solve the problem.

An obvious simplification can be obtained by fixing a priori n = 1. As can be seen, however, a true maximum is not attained by this method. But due to the small increase of the total cash flow (a few per cent) when using n = 6 instead of n = 1, this simplification can be worthwhile. It

can also be noted that by changing the value of n, it is possible to modify the well recovery to a considerable extent. This is illustrated by fig. 8, case B_3, and fig. 13, case B_2.

Two categories of results are shown from figs. 7 to 13.

1) <u>Constant output-equality cases</u> where

$$\sum_{j=1}^{m} q_{i,j}(t) = \text{constant (in arbitrary units)} \qquad 3.6$$

a) no limitations on individual flowrate

fig. 7 $n = 6$

fig. 12 $n = 12$

b) limitations on individual flowrate

fig. 11 $n = 6$

2) <u>Total bounded output-inequality cases</u> where

$$\sum_{j=1}^{m} q_{i,j}(t) \leq \text{constant (in arbitrary units)} \qquad 3.7$$

figs. 8, 9, 10 $n = 6$

fig. 13 $n = 12$

The results are shown with the following characteristics:

-- on the left side of the page, the economics based on the α-coefficient values

-- in the center of the page, the production profile

-- on the right side of the page, the total and individual output for the three wells

Four basic mechanisms have been observed:

A. <u>For the constant output-equality cases</u>, an improvement of the economics leads to an increase of the cash flow associated with a decrease of the recovery from each well.

B. For the total bounded output-inequality cases we fix a constant demand and follow the change of the optimum flowrates for different values of the constant. The interesting parameters are the decrease of the demand and the change of the economics.

Then:

-- for large values of the demand, all the wells have to flow to their maximum;

-- when decreasing the demand, the best wells are kept at their maximum value while the other wells are reduced conversely proportional to their productivity;

-- for a given value of the demand corresponding to a maximum constant demand for the whole period, the situation is completely modified;

-- as soon as the demand is below the previous value, the bad wells are fixed to their maxima and the final fit is made on the best well in reference to its productivity.

This procedure can be followed on figs. 8, 9, 10, and 13. On figs. 7 and 12, the decrease of the maximum value of a constant demand is shown to be a function of the different economics imposed on the wells.

C. Effects of upper and lower bounds are shown on fig. 11, which can be compared with fig. 7.

D. Production profile. During the entire period the well production has to satisfy a certain profile. The first steps of this profile are especially interesting.

The results shown can be considered as typical behavior for a three-well system where

well (1) is a low productivity well

well (2) is a medium productivity well

well (3) is a high productivity well

The different values of the economics must be sufficient to cover most of the actual cases.

Using an IBM 7094 computer and Share program number HOSCM3, the computing time was 2 seconds for each case when n = 6.

4 CONCLUSION

-- A good estimate of the optimum cash flow can be obtained with n = 10.

-- The optimum recovery of a well is very sensitive to the economics imposed on the system, as, for example, the lower and upper bounds restrictions.

-- The bad wells must produce at a flowrate proportionately higher than the best wells.

-- The production profile of the well is not obvious in most cases, but an accurate approach to the optimum profile can be obtained at moderate cost by using a linear programming model.

5 APPENDIX

5.1 THE NEYMAN-PEARSON LEMMA

5.11 Statement of the problem

We wish to minimize a linear functional of the form:

$$\int_0^1 f(\tau)\, a(\tau)d\tau \qquad 5.1101$$

over all functions $f(\tau)$ subject to the following constraints:

a) $0 \leq f(\tau) \leq 1, \quad 0 \leq \tau \leq 1$ 5.1102

b) $\int_0^1 f(\tau)b(\tau)d\tau \leq c$ 5.1103

where $a(\tau) \leq 0$ and $b(\tau) \geq 0$ are given functions, and $c \geq 0$ is a known constant.

5.12 Solution

The solution is determined as follows. Let set functions

$$E^- = E^-(\alpha'), \quad E = E(\alpha'), \quad E^+ = E^+(\alpha')$$

be defined for $-\infty < \alpha < +\infty$ as:

$$E^-(\alpha') = [\tau;\ a(\tau) < \alpha' b(\tau)]$$

$$E(\alpha') = [\tau;\ a(\tau) = \alpha' b(\tau)]$$

$$E^+(\alpha') = [\tau;\ a(\tau) > \alpha' b(\tau)]$$

Determine α_0' by the condition that α_0' be the supremum over all non-positive α' satisfying the inequality

$$\int_{E^-} b(\tau)d\tau \leq c \qquad 5.1104$$

and let

$$E^-(\alpha_0') = E_0^-$$

$$E\,(\alpha_0') = E_0$$

$$E^+(\alpha_0') = E_0^+$$

Then the set of minimizing functions f^* is given by:

a) $f^*(\tau) = 1$ on E_0^-

b) $f^*(\tau) = 0$ on E_0^+

c) $f^*(\tau) =$ arbitrary on E_0, satisfying only the condition

$$0 \leqq f^*(\tau) \leqq 1, \quad 0 \leqq \tau \leqq 1$$

and

$$\int_0^1 f^*(\tau)b(\tau)d\tau = c \qquad \text{if } \alpha_0' < 0$$

or the conditions

$$0 \leqq f^*(\tau) \leqq 1, \quad 0 \leqq \tau \leqq 1 \qquad \text{if } \alpha_0' = 0$$

5.13 Outline of the procedure

a) choose $\alpha' = \alpha_i' < 0$

b) find $E^-(\alpha_i') = [\tau;\ a(\tau) < \alpha_i b(\tau)]$

c) compute $A = \int_{E^-} b(\tau)d\tau$

d) if $A_{(\tau)} > c$, decrease α_i' and go to (a)

if $A_{(\tau)} < c$, increase α_i' and go to (a)

Optimal α_i' is α_0' such that for $\alpha_i' + \Delta\alpha_i$, $A_{(\tau)} > c$. See fig. 14.

5.2 WELL FUNCTION OF THE SINGLE WELL

The well function, defined as the pressure drop in a given well corresponding to a constant unit flowrate at time t, can be computed in many ways (see, for instance, ref. 2). Here we are interested in the analytical formulation of the well function corresponding to a radial system, and we show that such a function can be written as:

$$\boxed{\bar{\omega}_0(\theta) = \text{Log } R' - 2\sum_{n=1}^{4} \alpha_n e^{-\beta_n^2 \epsilon}} \qquad 5.201$$

where each term is defined below.

Radial flow of a homogeneous fluid into a single well

The well completely penetrates the reservoir and the fluid flows uniformly in all directions radiating to the axis of the well bore. The system has axial symmetry and we can describe the fluid pressure variations by the Laplace equation in cylindrical coordinates, namely:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = \frac{1}{K}\frac{\partial p}{\partial r} \qquad 5.202$$

with dimensionless variables as follows:

distances: $r' = \frac{r}{a}$

time: $\theta = \frac{Kt}{a^2}$

pressure: $\bar{\omega} = (P_G - P)\,\frac{2\pi hk}{\eta Q}$

Equation 5.202 is then reduced to:

$$\frac{\partial^2 \bar{\omega}}{\partial r'^2} + \frac{1}{r'} \frac{\partial \bar{\omega}}{\partial r'} = \frac{\partial \bar{\omega}}{\partial \theta} \qquad 5.203$$

Initial conditions

-- constant pressure in the whole reservoir at the start, or:

$$\theta = 0, \quad \bar{\omega} = 0 \qquad 5.204$$

-- boundary conditions:

constant flowrate from the well, or:

$$\left(\frac{\partial \bar{\omega}}{\partial r'}\right)_{r'=1} = -1 \quad \text{for any } \theta \qquad 5.205$$

no flowrate from the outside border and constant pressure at the reservoir limit, or:

$$\bar{\omega} = 0 \quad \text{for any } \theta \qquad 5.206$$

Analytical solution

The solution of the above system with the given initial and boundary conditions is such that:

$$\boxed{\bar{\omega}_1(\theta) = \text{Log } R' - 2 \sum_{n=1}^{\infty} \frac{J_0^2(\beta_n R') e^{-\beta_n^2 \theta}}{\beta_n^2 \left[J_1^2(\beta_n) - J_0^2(\beta_n R')\right]}} \qquad 5.207$$

where the β_n's are roots of:

$$Y_0(\beta_n R') J_1(\beta_n) - J_0(\beta_n R') Y_1(\beta_n) = 0 \qquad 5.208$$

If we notice that the last term of 5.207 decreases rapidly as the time increases, after a certain time:

$$\bar{\omega}_1(\theta) = \text{Log } R' \qquad 5.209$$

Equation 5.207 represents the unsteady state and equation 5.208 represents the steady state. $\bar{\omega}_1(\theta)$ is shown on fig. 15.

Approximations $\bar{\omega}_2(\theta)$, $\bar{\omega}_3(\theta)$ and $\bar{\omega}_4(\theta)$

On fig. 15 we can differentiate three zones, namely:

zone 1: the initial part of the curve,

zone 2: the central part presenting a straight semilog variation,

zone 3: the end portion with the asymptotic value Log R'.

These three zones are bounded by times θ_1 and θ_2 which will be considered further.

If we now vary R', we see on fig. 16 that all the curves representing $\bar{\omega}_1\theta$ are tangent to the curve $\bar{\omega}_2(\theta)$ for which the external radius is infinite. Physically this property is very easy to justify. Then the calculus shows that when the radius is infinite:

$$\boxed{\bar{\omega}_2(\theta) = \frac{4}{\pi^2}\int_0^\infty \frac{1 - e^{-u^2\theta}}{u^3\left[J_1^2(u) + Y_1^2(u)\right]}\,du} \qquad 5.210$$

and for values of θ larger than θ_1 this last expression converges to

$$\boxed{\bar{\omega}_3(\theta) = \frac{1}{2}(\text{Log } \theta + 0.80907)} \qquad 5.211$$

The semilogarithmic variation for the case of a reservoir of finite radius can be used up to a certain value θ_2 which will be considered further.

If we now assume that the well radius is null and the outside radius infinite, the solution is such that:

$$\boxed{\bar{\omega}_4(\theta) = \frac{1}{2} E_i\left(-\frac{1}{4\theta}\right)} \qquad 5.212$$

This last function can be of some use when we want to compute the well function in an interval such that $\theta \leq \theta_1$. Besides its obvious simplicity compared with $\bar{\omega}_1(\theta)$ or $\bar{\omega}_2(\theta)$, an interesting use of the $\bar{\omega}_0(\theta)$ function can be seen through its derivative when referring to the Neyman-Pearson method, for instance.

Interval for the use of $\bar{\omega}_3(\theta)$: θ_1 and θ_2

θ_1: the convergence of $\bar{\omega}_1(\theta)$ to $\bar{\omega}_3(\theta)$ is illustrated on fig. 17, where we can see that the approximation $\bar{\omega}_3(\theta)$ for $\bar{\omega}_1(\theta)$ is excellent as far as $\theta \geq 10^2$ and this limit is practically independent of R'.

θ_2: the value of θ_2 is defined as the limiting value of θ up to which we get an appropriate convergence between $\bar{\omega}_1(\theta)$ and $\bar{\omega}_3(\theta)$. If we fix a ratio of convergence such that:

$$\frac{\bar{\omega}_1(\theta) - \bar{\omega}_3(\theta)}{\bar{\omega}_1(\theta)} = 6/1000 \qquad 5.213$$

we get fig. 18, from where:

$$\operatorname{Log} \theta_2 = a \operatorname{Log} R' + b \qquad 5.214$$

which shows that θ_2 is dependent on R' under a simple formulation.

Calculation of α_n and β_n

α_n: from equation 5.207,

$$\alpha_n = \frac{J_0^2(\beta_n R')}{\beta_n^2\left[J_1^2(\beta_n) - J_0^2(\beta_n R')\right]} = \frac{1}{\beta_n^2\left[\dfrac{J_1^2(\beta_n)}{J_0^2(\beta_n R')} - 1\right]} \qquad 5.215$$

but from 5.208,

$$\frac{J_1^2(\beta_n)}{J_0^2(\beta_n R')} = \frac{Y_1^2(\beta_n)}{Y_0^2(\beta_n R')} \qquad 5.216$$

and for small values of β_n

$$\text{Log } Y_1(\beta_n) = - \text{Log } \beta_n + \beta_a$$

or

$$\left(Y_1\right)_x = \left(Y_1\right)_a \frac{a}{x} \qquad 5.217$$

when $R' > 100 \begin{cases} Y_1 \text{ is large, say} > 600 \\ Y_0(\beta R') \text{ is small, say} < 1 \end{cases}$

$$\boxed{\alpha_n = 2.46738 \left[Y_0(\beta_n R')\right]^2} \qquad 5.218$$

and for large values of n in order to satisfy 5.204, from where

$$\text{Log } R' = 2\sum_{n=1}^{\infty} \alpha_n \qquad 5.219$$

we get

$$\boxed{\alpha_n = \frac{1}{2n}\left(1 - \frac{1}{R'}\right)^n} \qquad n \geq 15 \qquad 5.220$$

which is practically independent of R' as is shown on fig. 19, when, for instance, R' is larger than 100, this being a reasonable assumption for a practical case.

$\underline{\beta_n}$: the β_n's shown on fig. 20 are the roots of 5.208 and easy to compute from:

$$\frac{J_1(\beta_n)}{Y_1(\beta_n)} = \frac{J_0(\beta_n R')}{Y_0(\beta_n R')} \qquad 5.221$$

When $R' > 100$, β_n is small and $J_1(0) = 0$; then $J_1(\beta_n) \simeq 0$, and

$$\boxed{J_0(\beta_n R') = 0} \qquad 5.222$$

As soon as n is larger than 15, the positive roots of 5.222 are given approximately by:

$$\boxed{\beta_n = \frac{\pi}{R'}\left(n - \frac{1}{4}\right)} \qquad n \geq 15 \qquad 5.223$$

The following table summarizes the values of α_n and $\beta_n R'$ ($R' \geq 100$) obtained from the above considerations.

Table 1

n	α_n	$\beta_n R'$	n	α_n	$\beta_n R'$	n	α_n	$\beta_n R'$
1	0.6415	2.405	11	0.0465	33.78	21	0.0241	65.18
2	0.2834	5.520	12	0.0425	36.92	22	0.0230	68.33
3	0.1812	8.654	13	0.0392	40.06	23	0.0220	71.47
4	0.1331	11.79	14	0.0364	43.20	24	0.0211	74.62
5	0.1051	14.93	15	0.0339	46.34	25	0.0202	77.76
6	0.0869	18.07	16	0.0317	49.48	26	0.0194	80.90
7	0.0740	21.21	17	0.0299	52.62	27	0.0187	84.04
8	0.0645	24.35	18	0.0281	55.77	28	0.0180	87.18
9	0.0571	27.50	19	0.0267	58.90	29	0.0174	90.32
10	0.0512	30.63	20	0.0253	62.05	30	0.0168	93.46

The asymptotic well function $\bar{\omega}_0(\theta)$

In some particular cases, and for a specific interval of time, the use of $\bar{\omega}_2(\theta)$, $\bar{\omega}_3(\theta)$, or $\bar{\omega}_4(\theta)$ can simplify the problem. This is especially observed when using the function $\bar{\omega}_3(\theta)$ between θ_1 and θ_2. This last interval is sufficiently large in actual so that a semilogarithm variation is justified almost a priori for a given well. Nevertheless, it seems interesting to refer to a well function covering the complete time scale. Such a function can be estimated from the computed coefficients α_n and β_n. Only finite values of n will be used in order to simplify the formulation without altering its effective accuracy. Due to its property this function can be named the asymptotic well function $\bar{\omega}_0(\theta)$.

A computation of $\bar{\omega}_0(\theta)$ using the α_n and β_n coefficients from table 1 shows that the results reproduce those presented in reference 7 and that for $R' > 100$, the convergence of the number of terms is the following:

$$\theta = 1 \qquad n = 81$$

$$\theta = 10 \qquad n = 29$$

$$\theta = 10^2 \qquad n = 10$$

$$\theta = 10^3 \qquad n = 4$$

with a rapid convergence to one for larger values of θ depending on the actual value of R'.

It is then suggested that only the first four values of n be introduced for obtaining a good estimation of the well function of a single well located at the center of a circular reservoir.

NOMENCLATURE

a: well radius

$a(\tau)$: function (see 2.208)

$\bar{b}$: coefficient

$b(\tau)$: function (see 2.212)

$c(\tau)$: function (see 2.213)

$E_i(-t)$: exponential integral

$f_j(\tau)$: modified pressure drop at well j for modified time τ

$F_j(t)$: pressure drop at well j for a constant flowrate during time t

h: reservoir thickness

i: integer = 1, 2, ..., n, n + 1

$I_1(\tau)$: function (see 2.210)

j: integer = 1, 2, ..., m - 1, m

k: reservoir permeability

K: hydraulic diffusivity

m: number of wells in the multiwell system

n: number of equal intervals in period T

N: integer = 1, ..., n

$P_{0,j}$: initial pressure at well j

$P_{i,j}$: pressure at well j, time t_i

$P_j(t)$: pressure at well j, time t

$P_{x,j}$: pressure at well j, time x

$q_{i,j}$: discrete flowrate at well j, time t_i

$q_j(t)$: continuous flowrate at well j, time t

$q_j'(t)$: derivative of flowrate at well j, time t

$q_j max$: maximum flowrate allowed on well j

$q_j min$: minimum flowrate allowed on well j

$q_{x,j}$: flowrate at well j, time t_x

$q_{x,j}'$: derivative of flowrate at well j, time t_x

$Q(t)$: total flowrate of the reservoir at time t

Q_{T_m} : total production of the well m

r': reduced well radius

R: reservoir radius

R_j: reserves for well j

t: time $0 \leq t \leq T$

T: total period under consideration

t_i: time at which the flowrate q_i starts $0 \leq t_i \leq T$

t_x: time $0 \leq t_x \leq T$

y: integer = 1, 2, ..., n

$Y_{01}Y_{11}, J_0, J_1$: Bessel functions

α: function

α_n: roots (see 5.220)

β: oil compressibility

β_n: roots (see 5.223)

γ: coefficient

δ_j: actual pressure drop allowed at well j

Δ_j: modified pressure drop allowed at well j

Δ_t: interval of time

θ: reduced time

λ: coefficient

μ: oil viscosity

τ: modified time

τ_i^*: average modified time

ω: reservoir porosity

$\overline{\omega}(\theta)$: well function

LIST OF FIGURES

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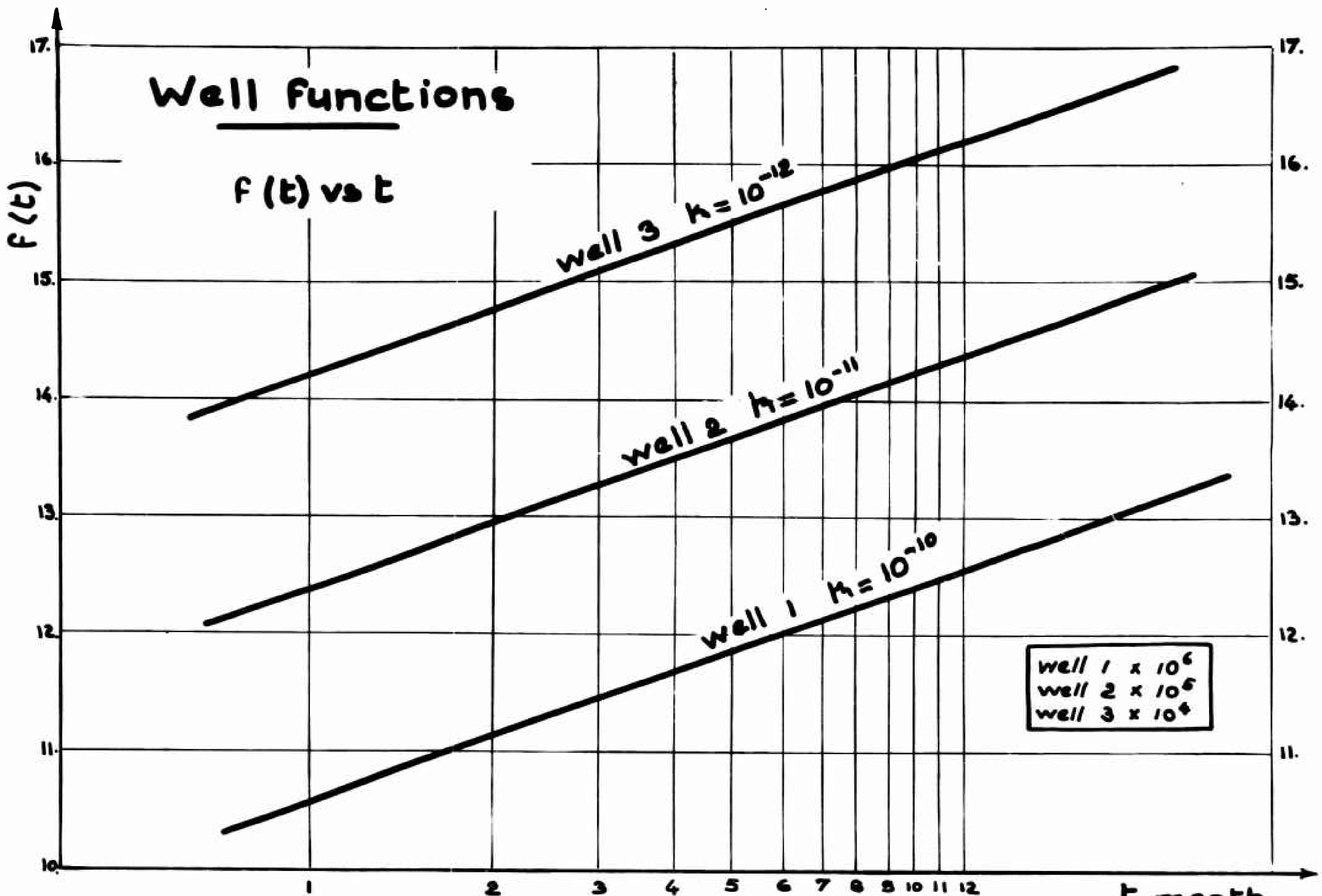
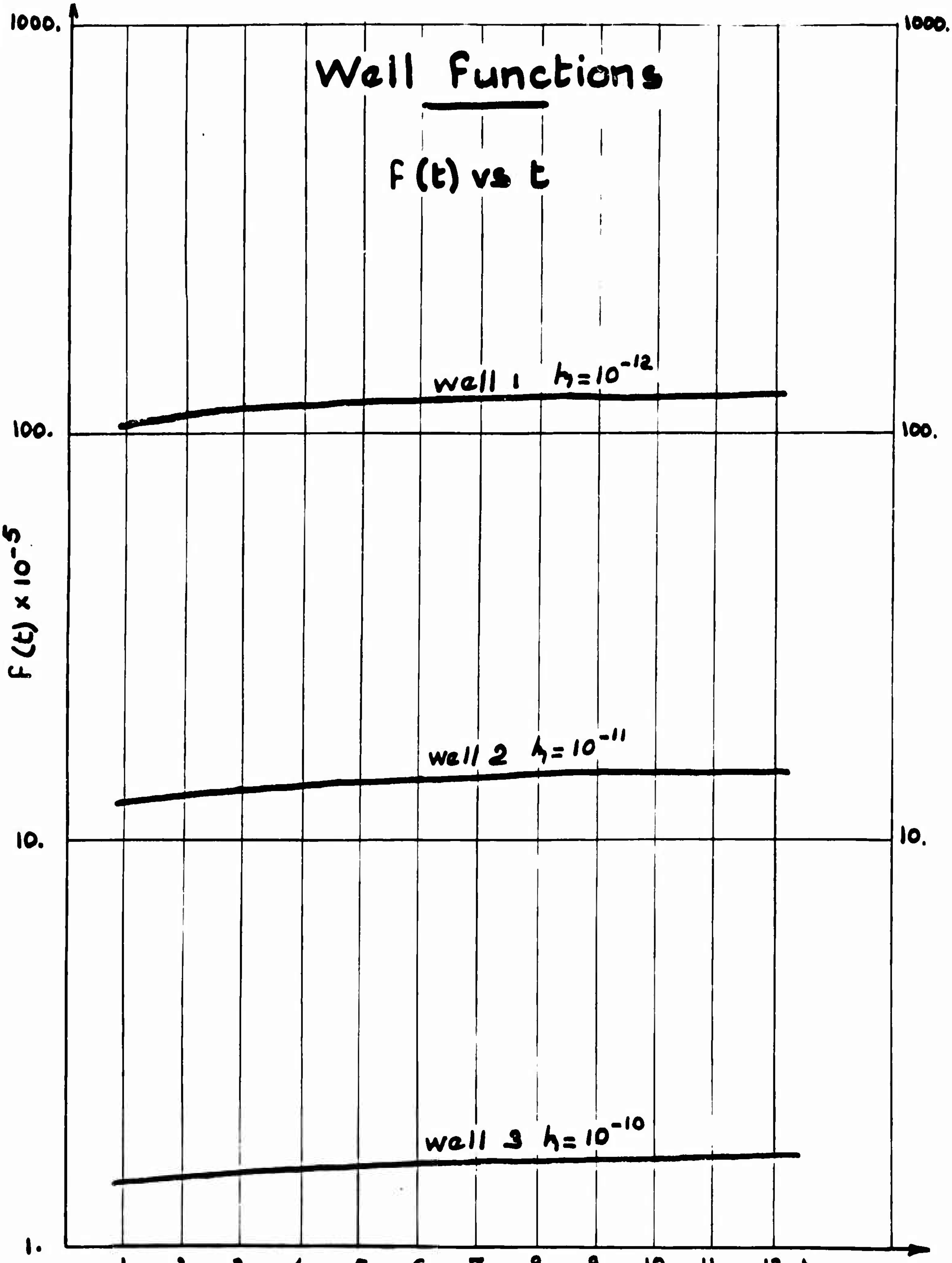
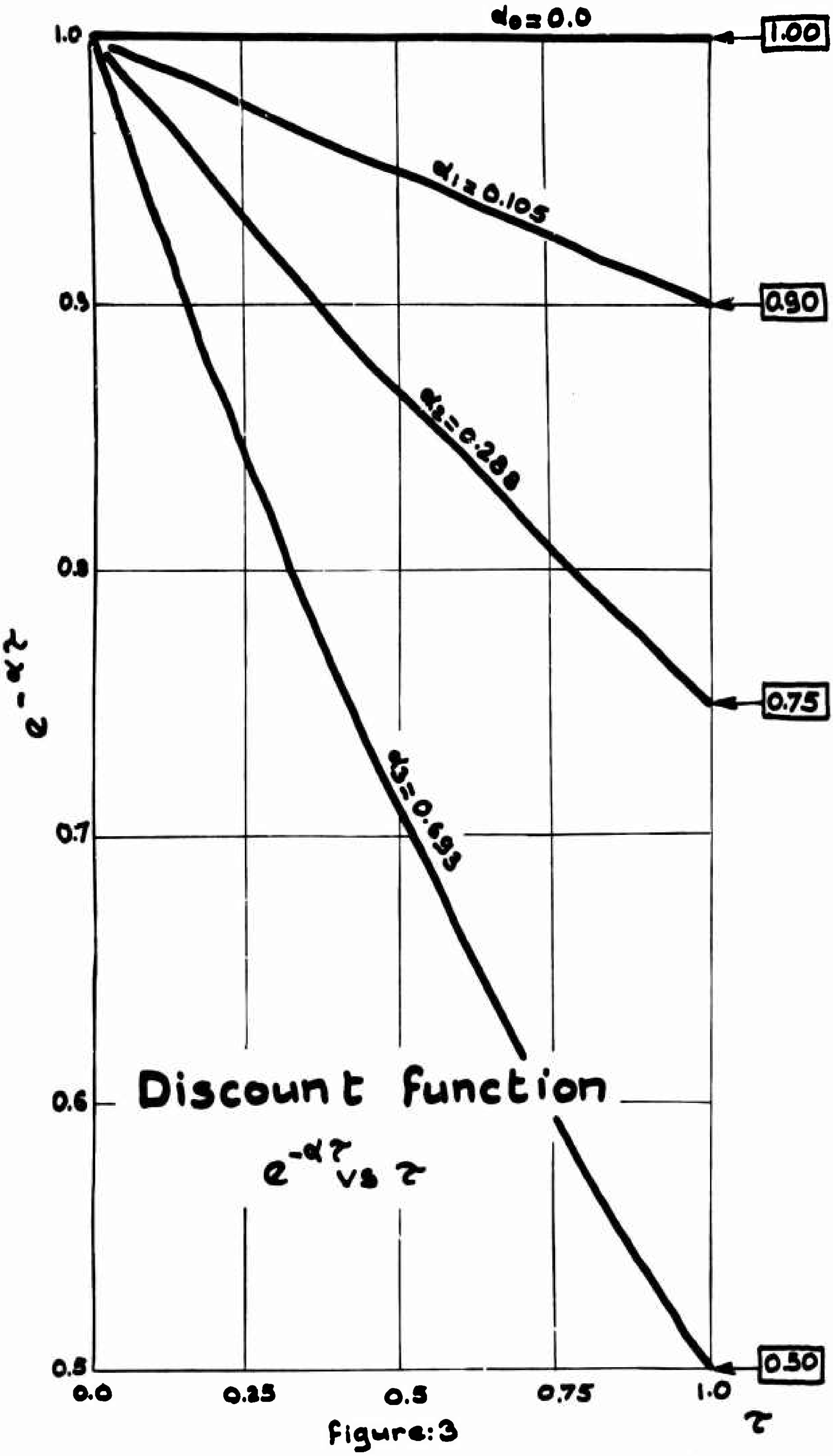

Well functions
f (t) vs t
f (t)
17.
16.
15.
14.
13.
12.
11.
10.
well 3 k = 10^-12
well 2 k = 10^-11
well 1 k = 10^-10
well 1 x 10^6
well 2 x 10^5
well 3 x 10^4
1
2
3
4
5
6
7
8
9
10
11
12
t month

Figure: 1


Well Functions
f (t) vs t
1000.
100.
10.
1.
f (t) x 10^-5
well 1 h=10^-12
well 2 h=10^-11
well 3 h=10^-10
1 2 3 4 5 6 7 8 9 10 11 12
t month
Figure: 2



$\alpha_0 = 0.0$
$\alpha_1 = 0.105$
$\alpha_2 = 0.288$
$\alpha_3 = 0.693$
1.00
0.90
0.75
0.50
1.0
0.9
0.8
0.7
0.6
0.5
0.0
0.25
0.5
0.75
1.0
τ
$e^{-\alpha\tau}$
Discount function
$e^{-\alpha\tau}$ vs τ

Figure: 3

Single well matrix n=12

(limitations on individual flowrates)

od_1	od_2	od_3	od_4	x_{01}^1	x_{02}^1	x_{03}^1	x_{04}^1	x_{05}^1	x_{06}^1	x_{07}^1	x_{08}^1	x_{09}^1	x_{10}^1	x_{11}^1	x_{12}^1	y_{01}	y_{02}	y_{03}	y_{04}	y_{05}	y_{06}	y_{07}	y_{08}	y_{09}	y_{10}	y_{11}	y_{12}	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}	V_{12}	b
-1.				-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.																																					
	-1.			-.99	-.98	-.97	-.96	-.96	-.95	-.94	-.94	-.93	-.92	-.91	-.90																																					
		-1.		-.99	-.96	-.94	-.92	-.89	-.87	-.86	-.85	-.82	-.79	-.78	-.76																																					
			-1.	-.97	-.92	-.87	-.82	-.77	-.73	-.69	-.65	-.61	-.58	-.54	-.51																																					
				105.6												1.																																				1000
				5.516	105.6												1.																																			1000
				3.226	5.516	105.6												1.																																		1000
				2.291	3.226	5.516	105.6												1.																																	1000
				1.775	2.291	3.226	5.516	105.6												1.																																1000
				1.449	1.775	2.291	3.226	5.516	105.6												1.																															1000
				1.227	1.449	1.775	2.291	3.226	5.516	105.6												1.																														1000
				1.065	1.227	1.449	1.775	2.291	3.226	5.516	105.6												1.																													1000
				0.937	1.065	1.227	1.449	1.775	2.291	3.226	5.516	105.6												1.																												1000
				0.837	0.937	1.065	1.227	1.449	1.775	2.291	3.226	5.516	105.6												1.																											1000
				0.758	0.837	0.937	1.065	1.227	1.449	1.775	2.291	3.226	5.516	105.6												1.																										1000
				0.694	0.758	0.837	0.937	1.065	1.227	1.449	1.775	2.291	3.226	5.516	105.6												1.																									1000
				1.																								1.																								8.5
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														1.																								1.														8.5
															1.																								1.													8.5
				1.																																				-1.												6.0
					1.																																				-1.											6.0
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													1.																																				-1.			6.0
														1.																																				-1.		6.0
															1.																																				-1.	6.0

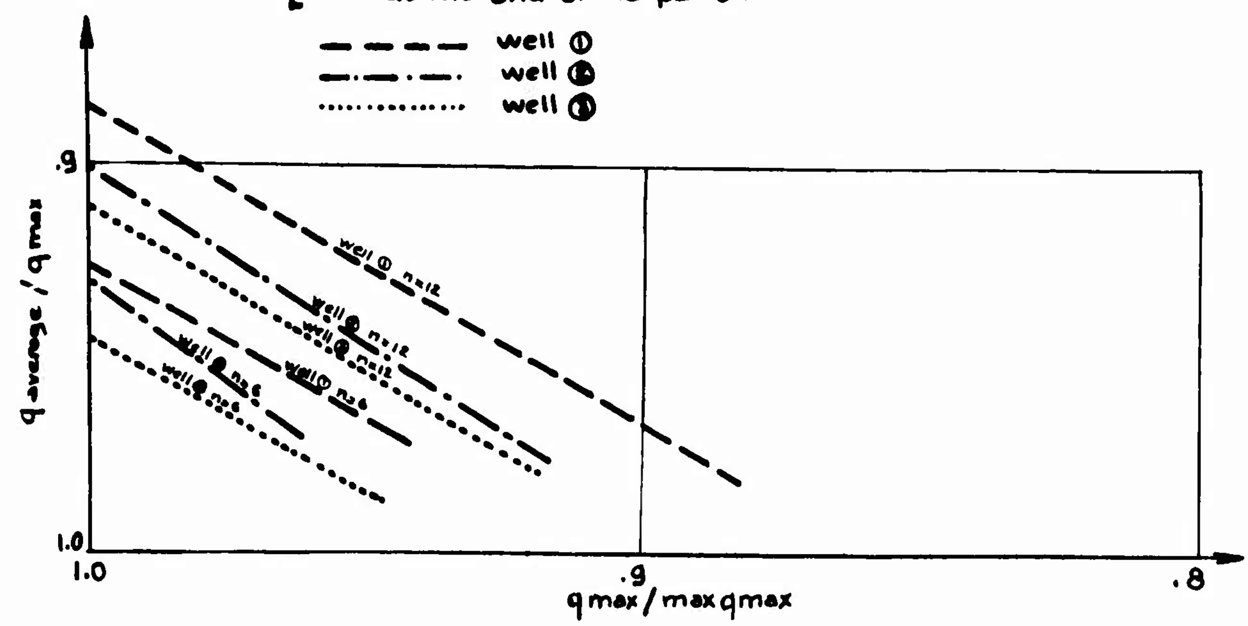
figure : 4

Single well

Flowrat-

(qmax ≤ maxqmax)

independent of α
maxi at the beginning of the period
mini at the end of the period



Profit: (qmax ≥ maxqmax)

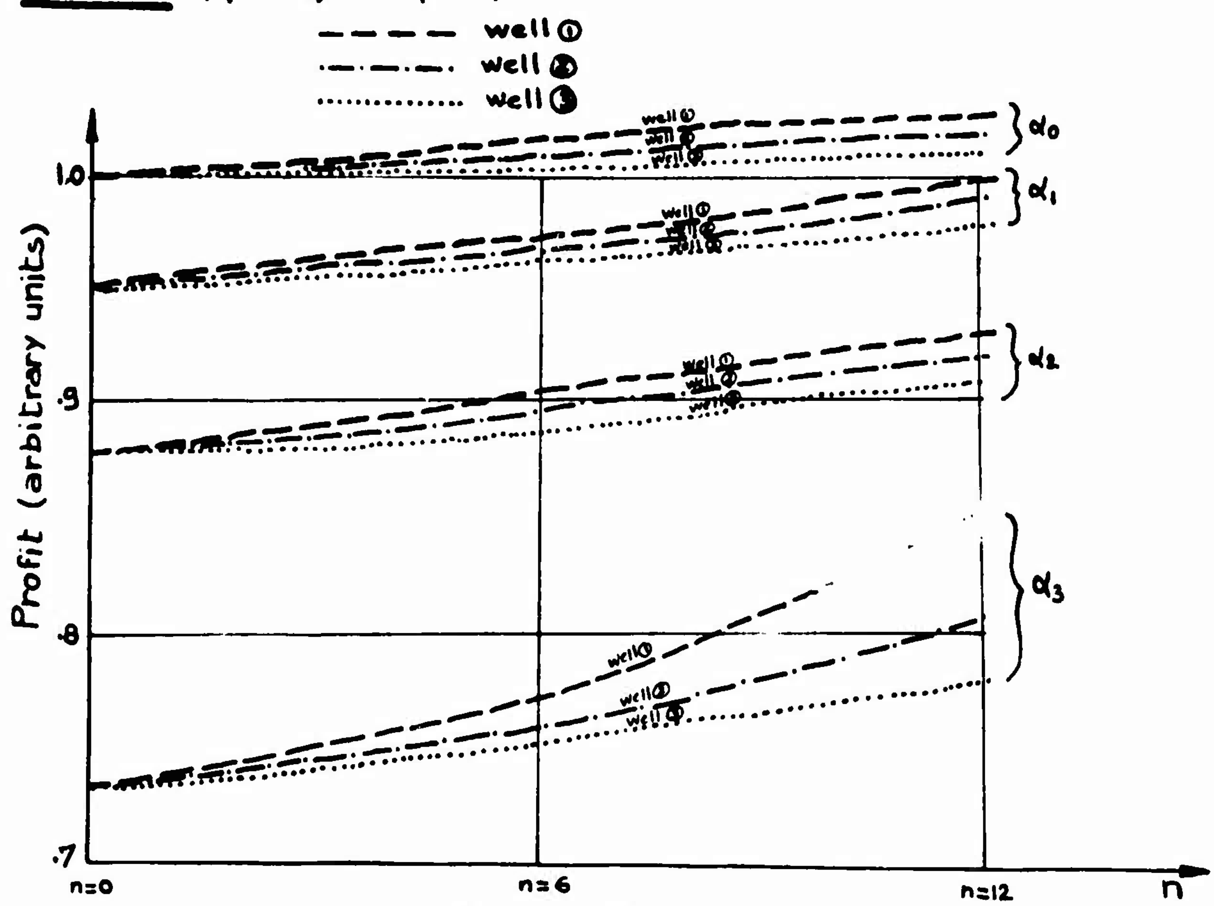


Figure: 5

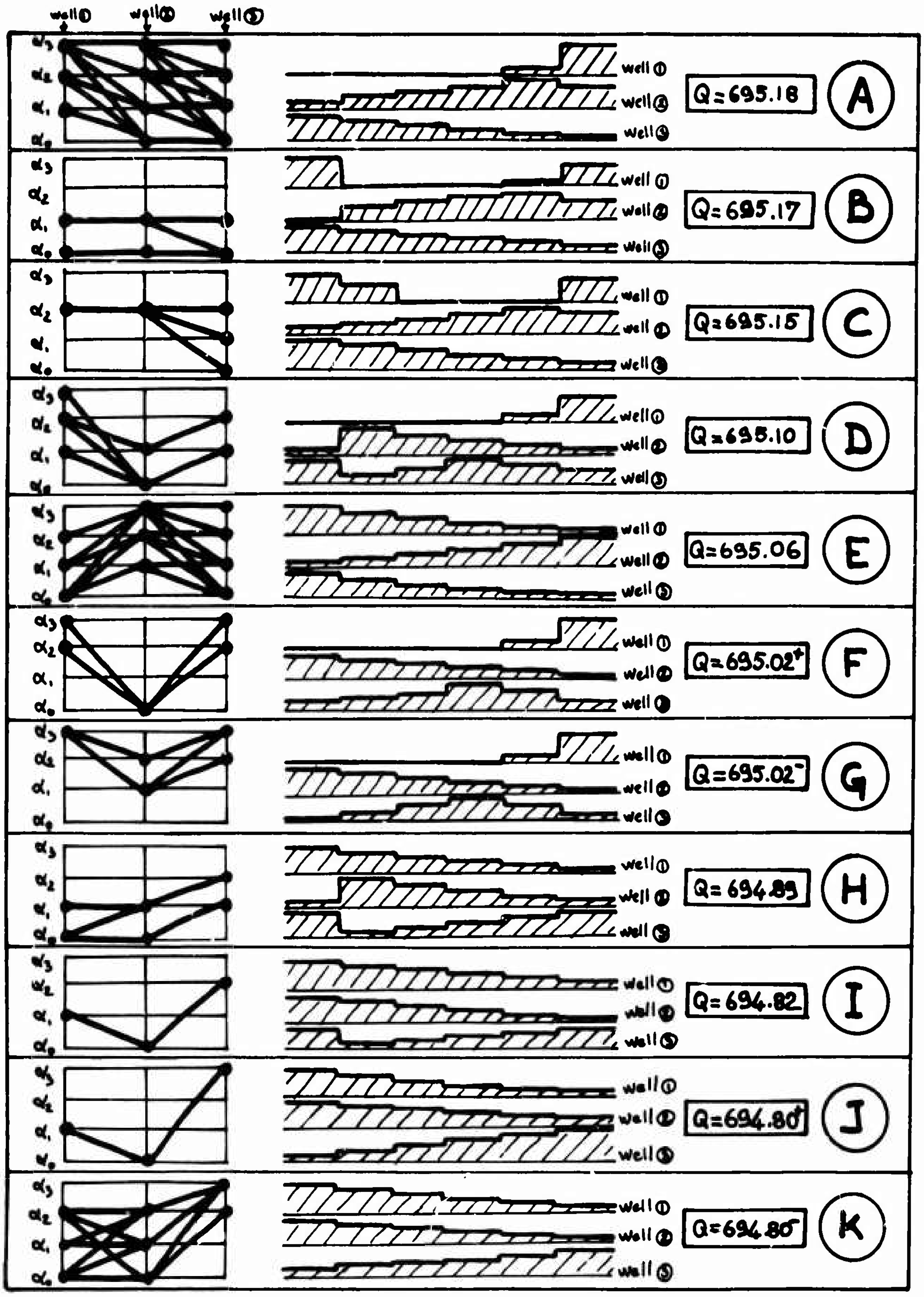
Multiwell matrix - Equality cases n=6

(no limitations on individual flowrates)

Qw31	X011	X031	X051	X071	X091	X111	X012	X032	X052	X072	X092	X112	X013	X033	X053	X073	X093	X113	Y011	Y031	Y051	Y071	Y091	Y111	Y012	Y032	Y052	Y072	Y092	Y112	Y013	Y033	Y053	Y073	Y093	Y113	Q	ΔP
-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.	-1.																				
	111.2																		1.0																			1000.
	5.517	111.2																		1.0																		1000.
	3.224	5.517	111.2																		1.0																	1000.
	2.292	3.224	5.517	111.2																		1.0																1000.
	1.774	2.292	3.224	5.517	111.2																		1.0															1000.
	1.452	1.774	2.292	3.224	5.517	111.2																		1.0														1000.
							12.95																		1.0													1000.
							0.552	12.95																		1.0												1000.
							0.322	0.552	12.95																		1.0											1000.
							0.229	0.322	0.552	12.95																		1.0										1000.
							0.177	0.229	0.322	0.552	12.95																		1.0									1000.
							0.145	0.177	0.229	0.322	0.552	12.95																		1.0								1000.
													1.478																		1.0							1000.
													0.055	1.478																		1.0						1000.
													0.032	0.055	1.478																		1.0					1000.
													0.028	0.032	0.055	1.478																		1.0				1000.
													0.018	0.028	0.032	0.055	1.478																		1.0			1000.
													0.014	0.018	0.028	0.032	0.055	1.478																		1.0		1000.
	1.0						1.0						1.0																								-1.	
		1.0						1.0						1.0																							-1.	
			1.0						1.0						1.0																						-1.	
				1.0						1.0						1.0																					-1.	
					1.0						1.0						1.0																				-1.	
						1.0						1.0						1.0																			-1.	

figure: 6

Total constant output - Equality cases n=6



Economics   Production profile   Total and Individual output

Figure: 7

Total bounded output - Inequality cases n=6

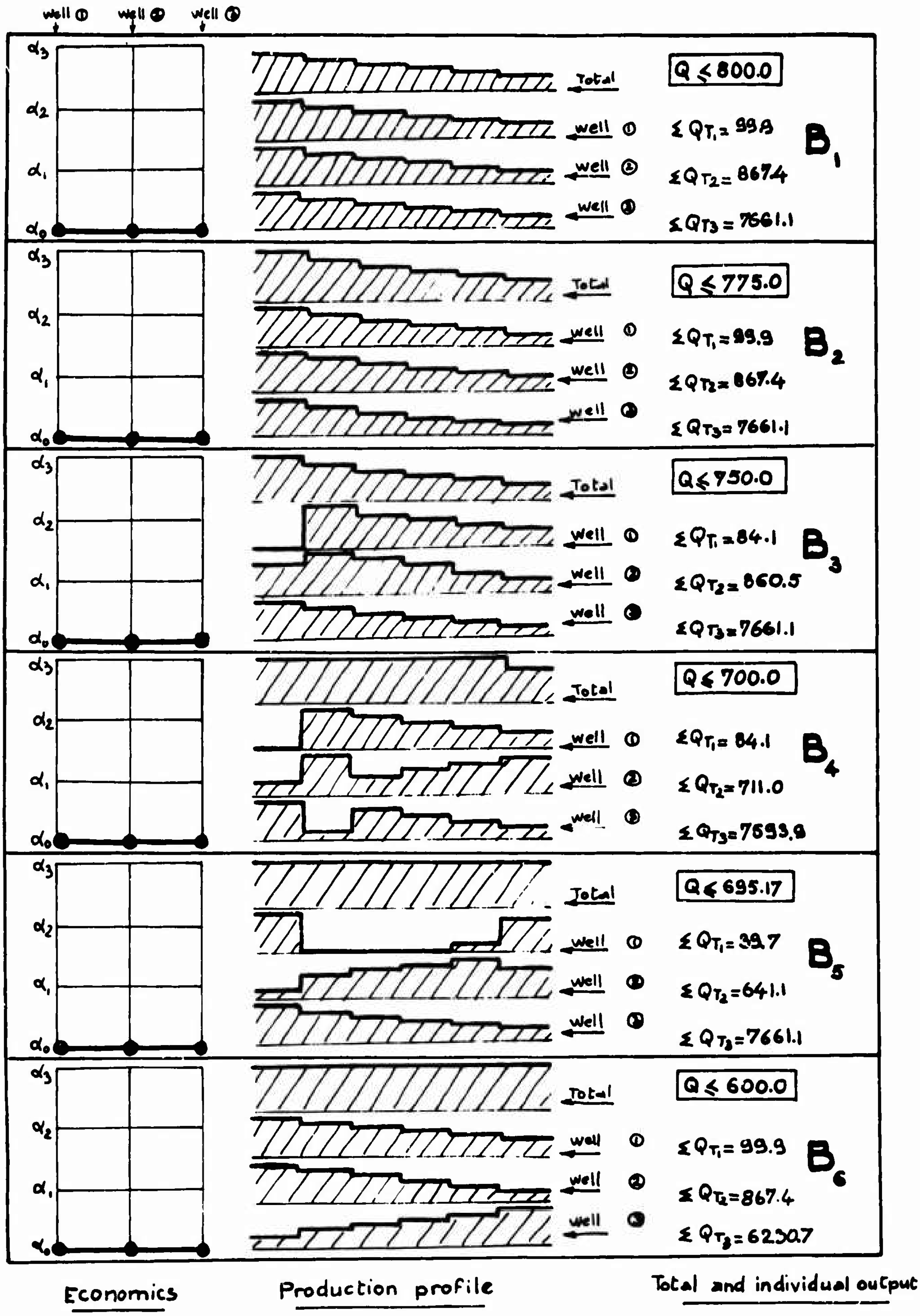


Figure: 8

Total bounded output - Inequality cases n=6

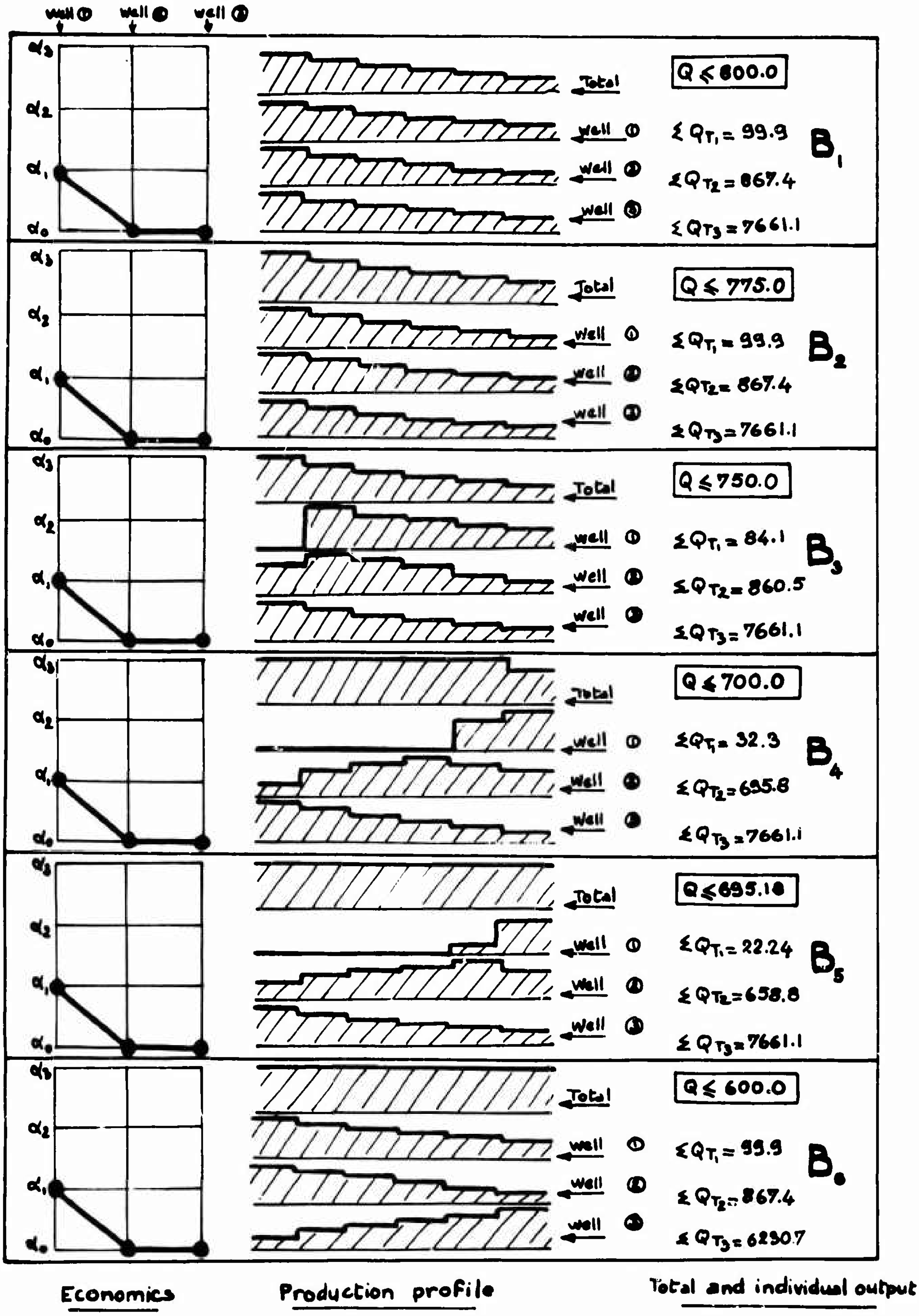


figure: 9

Total bounded output - Inequality cases n=6

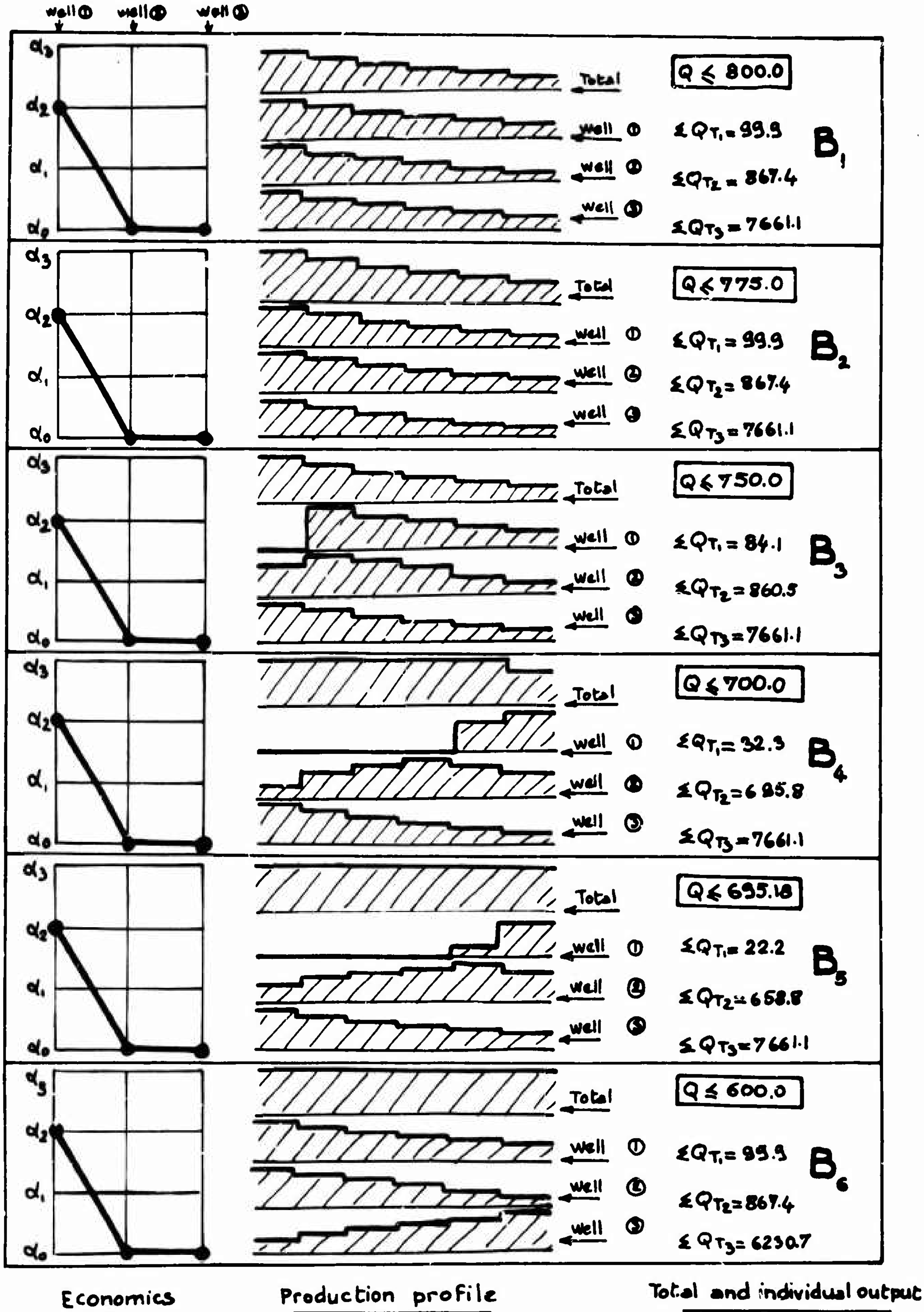


figure: 10

Total constant output - Equality cases n=6
(limitations on individual flowrates)

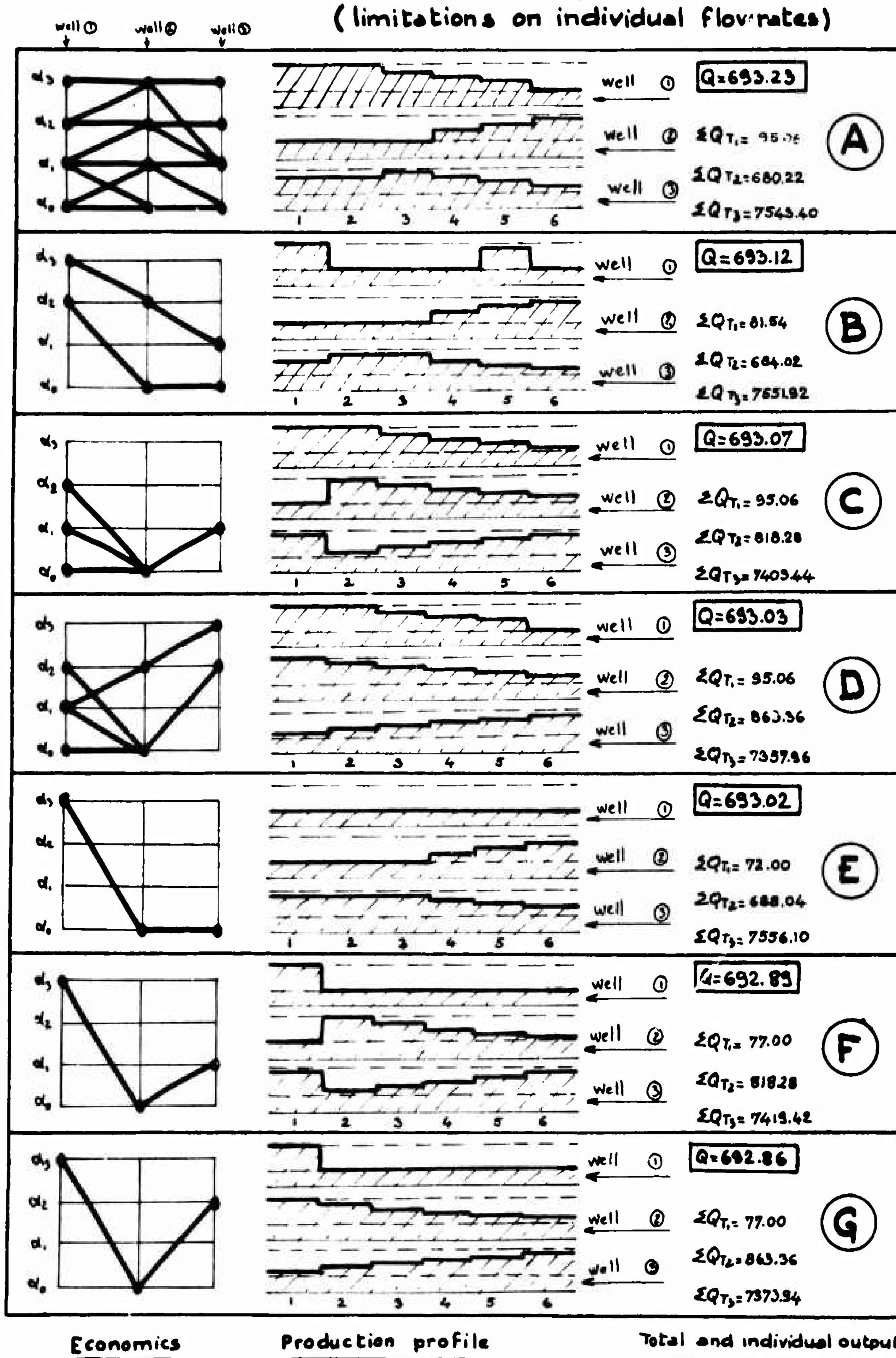


figure: 11

Total constant output - Equality cases n=12

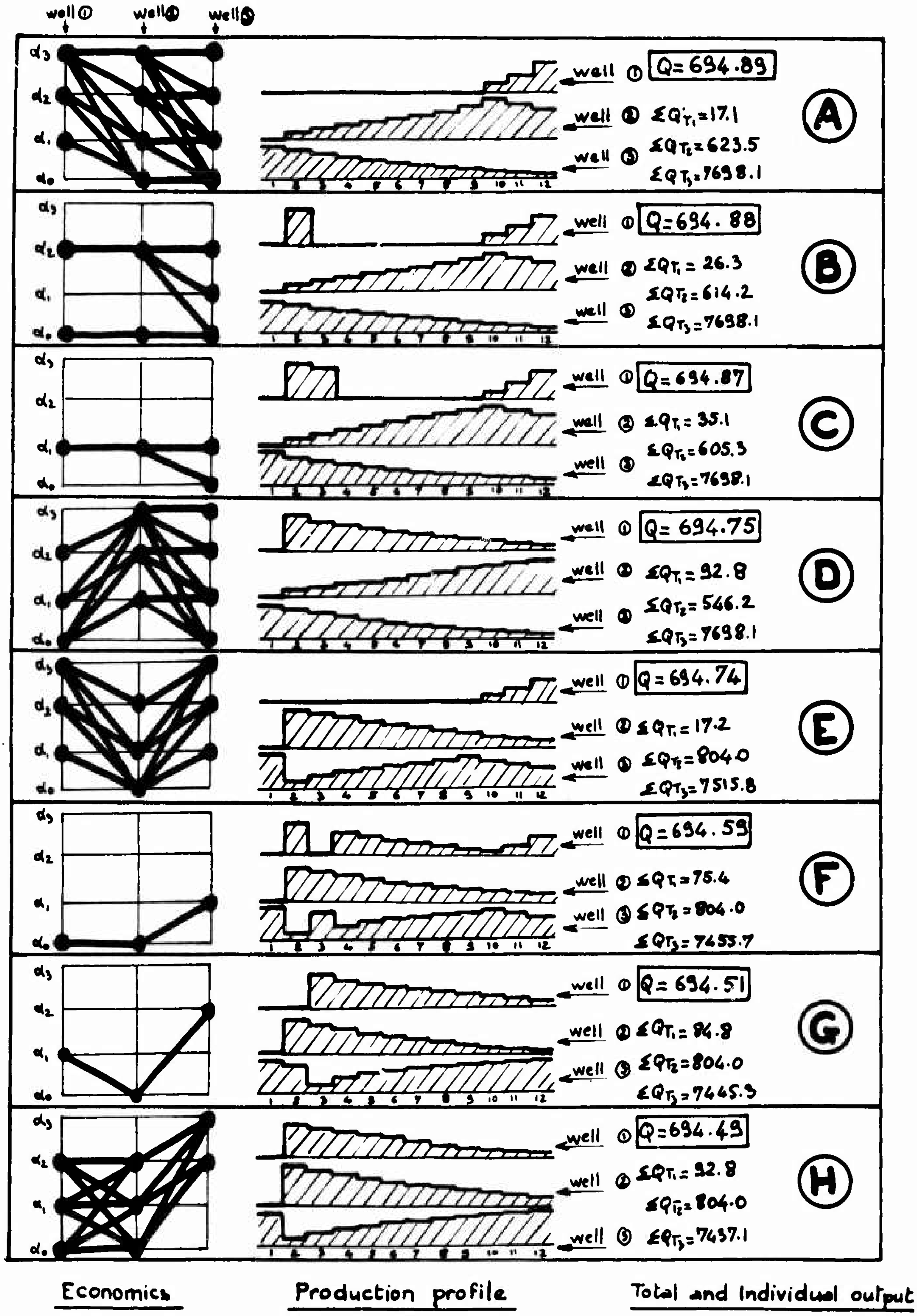
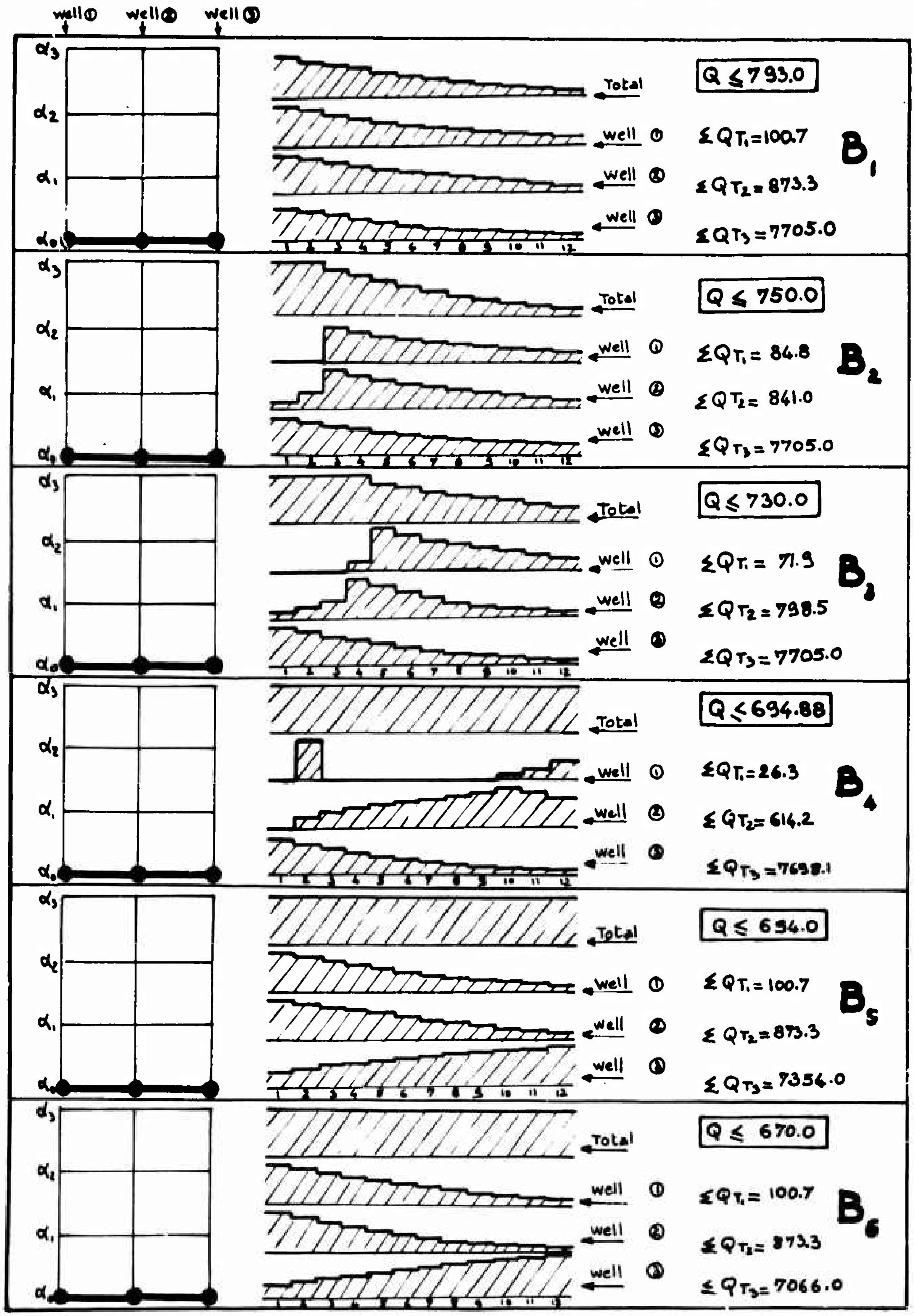


Figure: 12

Total bounded output - Inequality cases n=12



Economics Production profile Total and individual output

figure: 13

Neyman-Pearson lemma

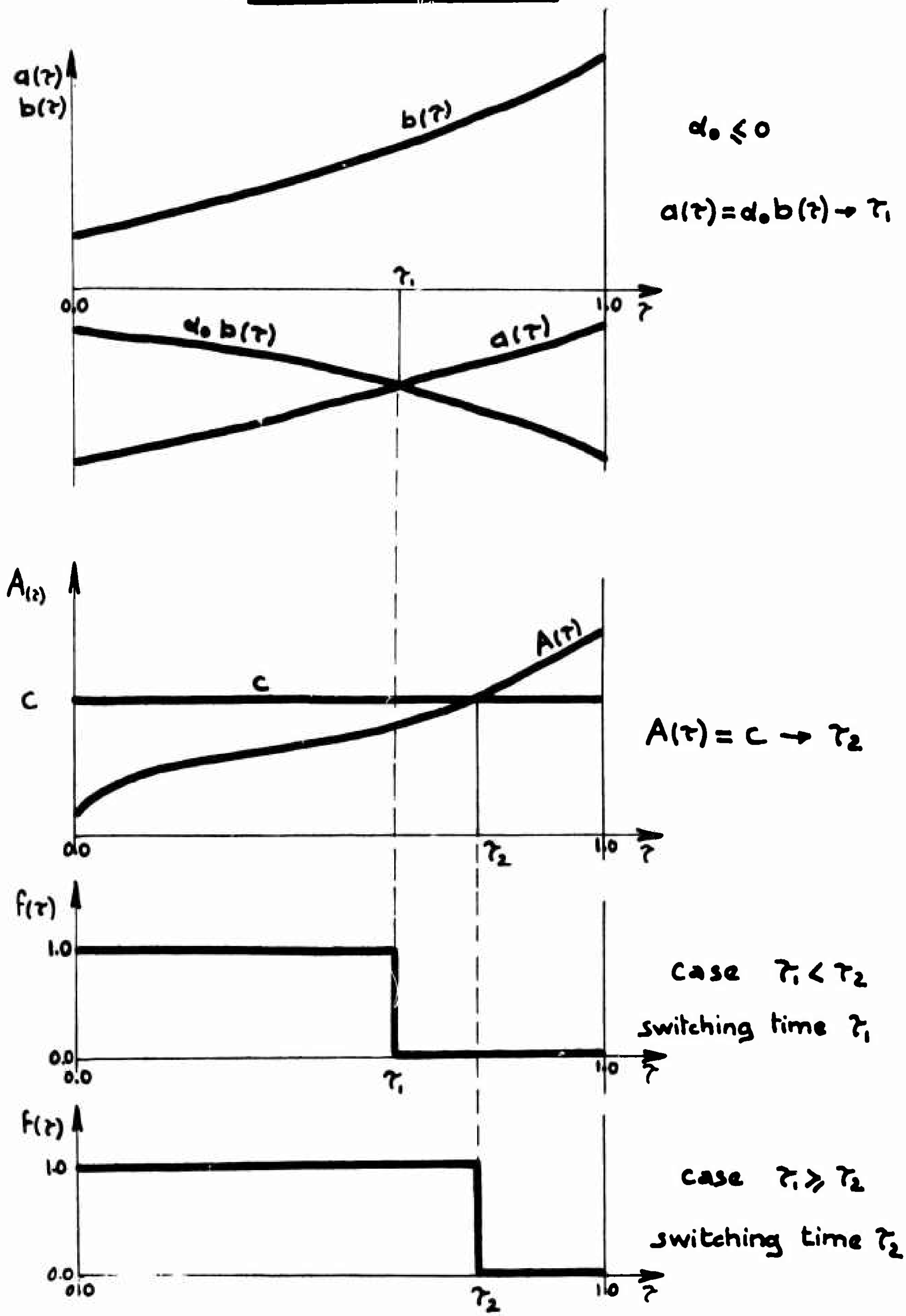
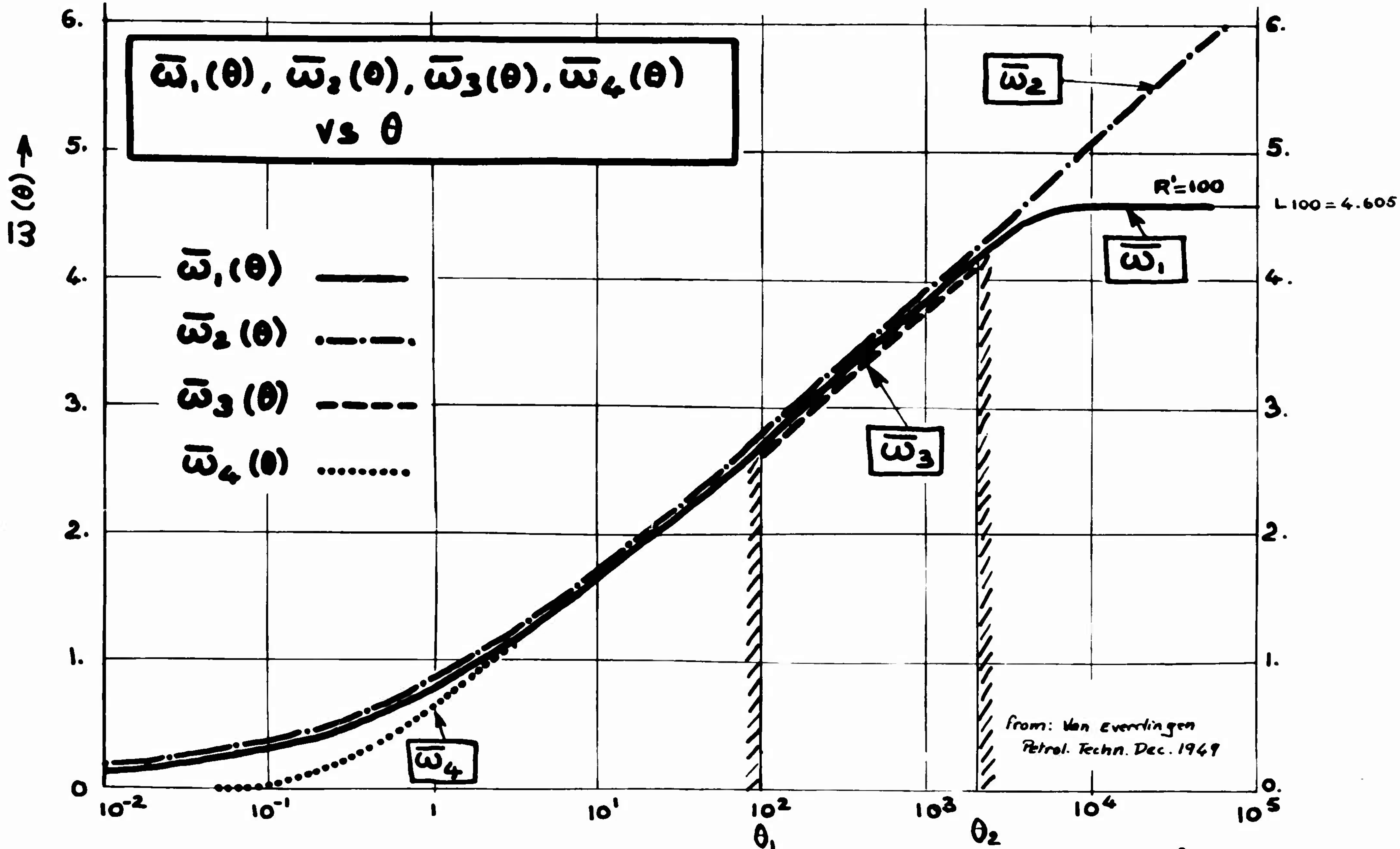


figure: 14



$\overline{\omega}_1(\theta)$, $\overline{\omega}_2(\theta)$, $\overline{\omega}_3(\theta)$, $\overline{\omega}_4(\theta)$ vs θ
$\overline{\omega}_1(\theta)$
$\overline{\omega}_2(\theta)$
$\overline{\omega}_3(\theta)$
$\overline{\omega}_4(\theta)$
$\overline{\omega}(\theta)$
$\overline{\omega}_1$
$\overline{\omega}_2$
$\overline{\omega}_3$
$\overline{\omega}_4$
$R' = 100$
$L\,100 = 4.605$
θ_1
θ_2
θ
from: Van Everdingen
Petrol. Techn. Dec. 1949

figure : 15

$\overline{\omega}_1(\theta)$ vs θ $3.0 \leq R' \leq 1000.$

$\overline{\omega}_2(\theta)$ vs θ $R' = \infty$

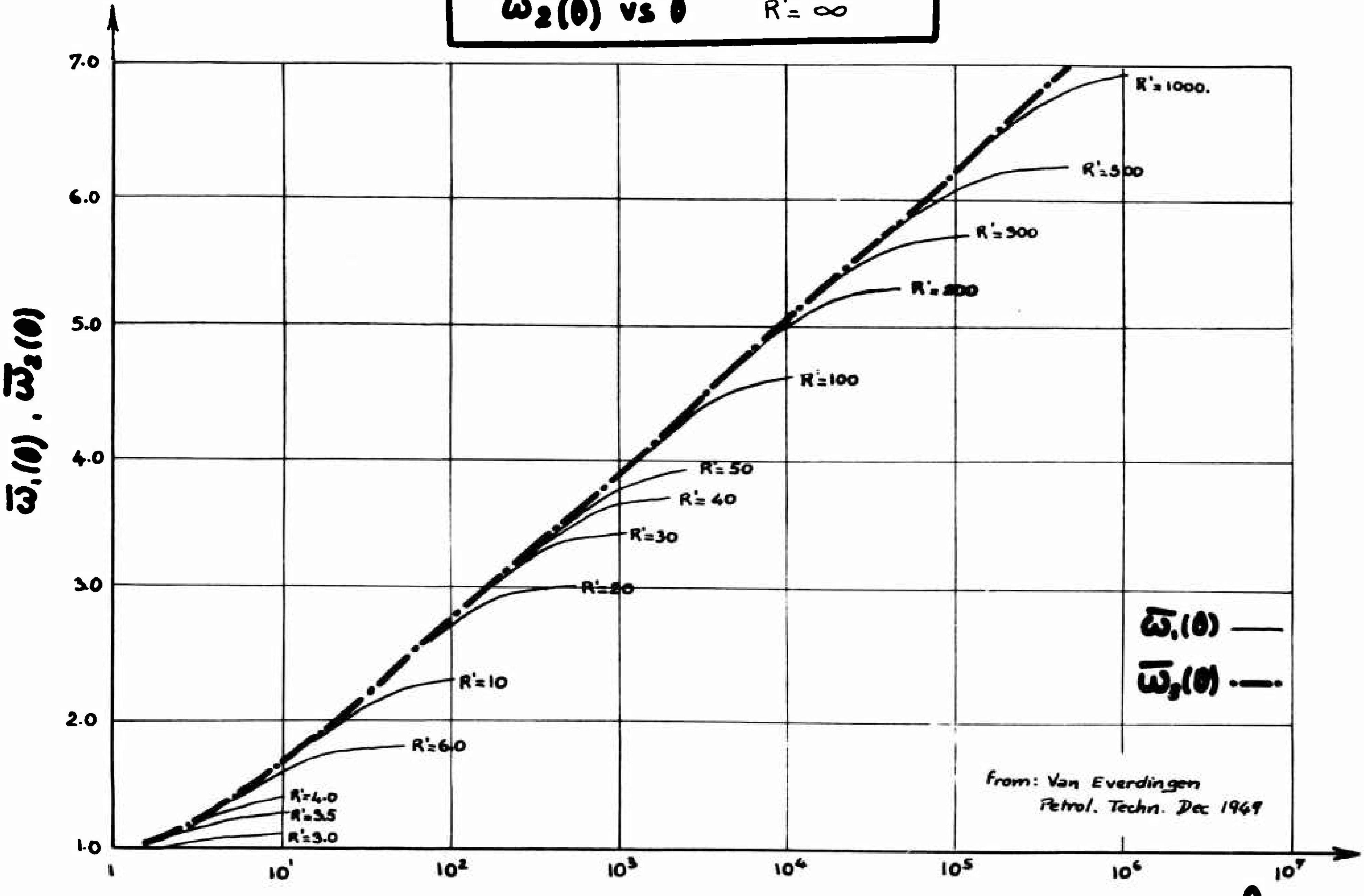
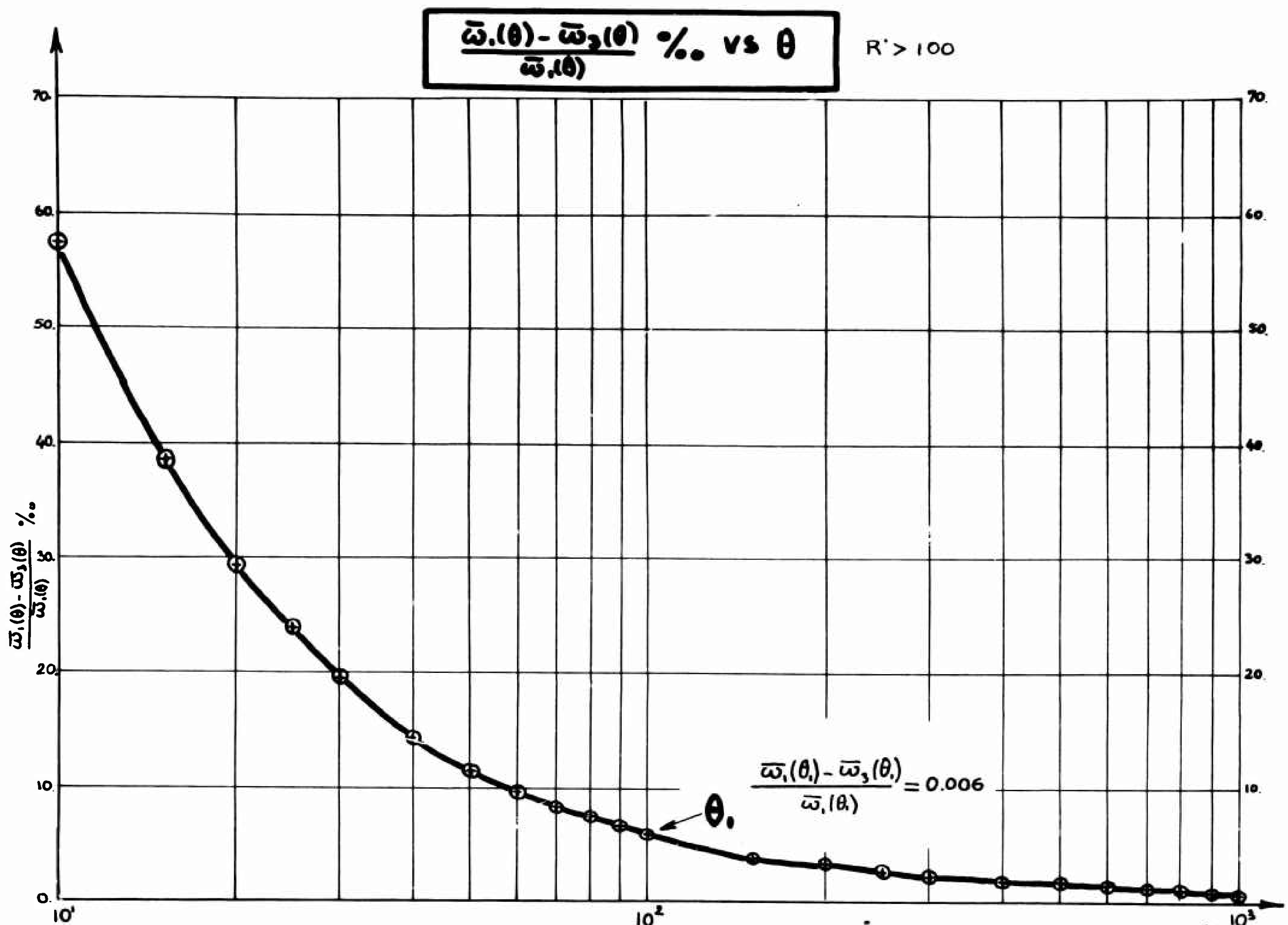


Figure : 16


$\frac{\bar{\omega}_1(\theta)-\bar{\omega}_3(\theta)}{\bar{\omega}_1(\theta)}$ ‰ vs θ
R' > 100
$\frac{\bar{\omega}_1(\theta_1)-\bar{\omega}_3(\theta_1)}{\bar{\omega}_1(\theta_1)} = 0.006$
θ_1
$\frac{\bar{\omega}_1(\theta)-\bar{\omega}_3(\theta)}{\bar{\omega}_1(\theta)}$ ‰
θ
0.
10.
20.
30.
40.
50.
60.
70.
10^1
10^2
10^3
figure: 17

θ_2 vs R'

$$\frac{\overline{\omega}_1(\theta) - \overline{\omega}_3(\theta)}{\overline{\omega}_1(\theta)} = cst$$

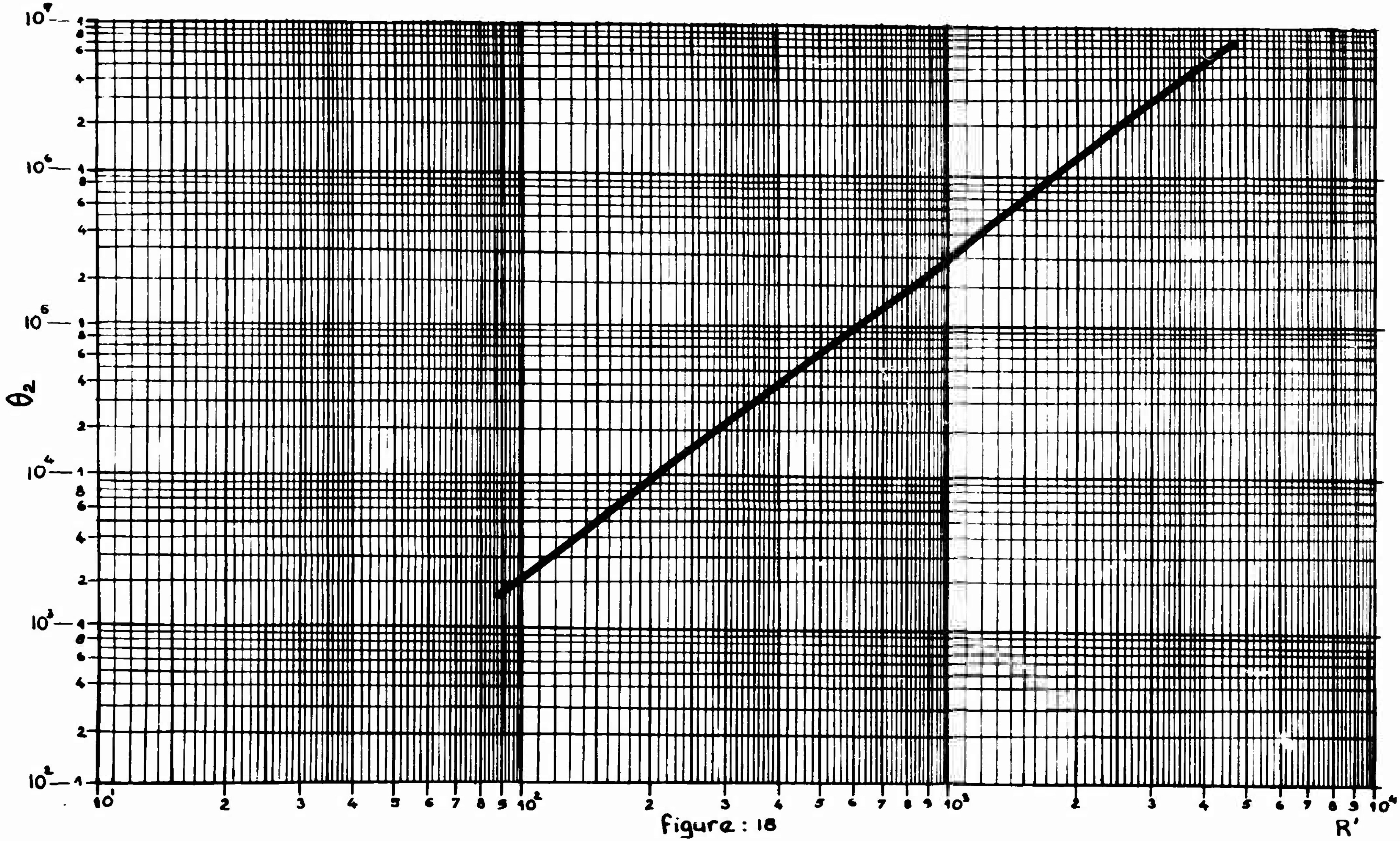


figure : 18

α_n vs n

$R' \geqslant 100$

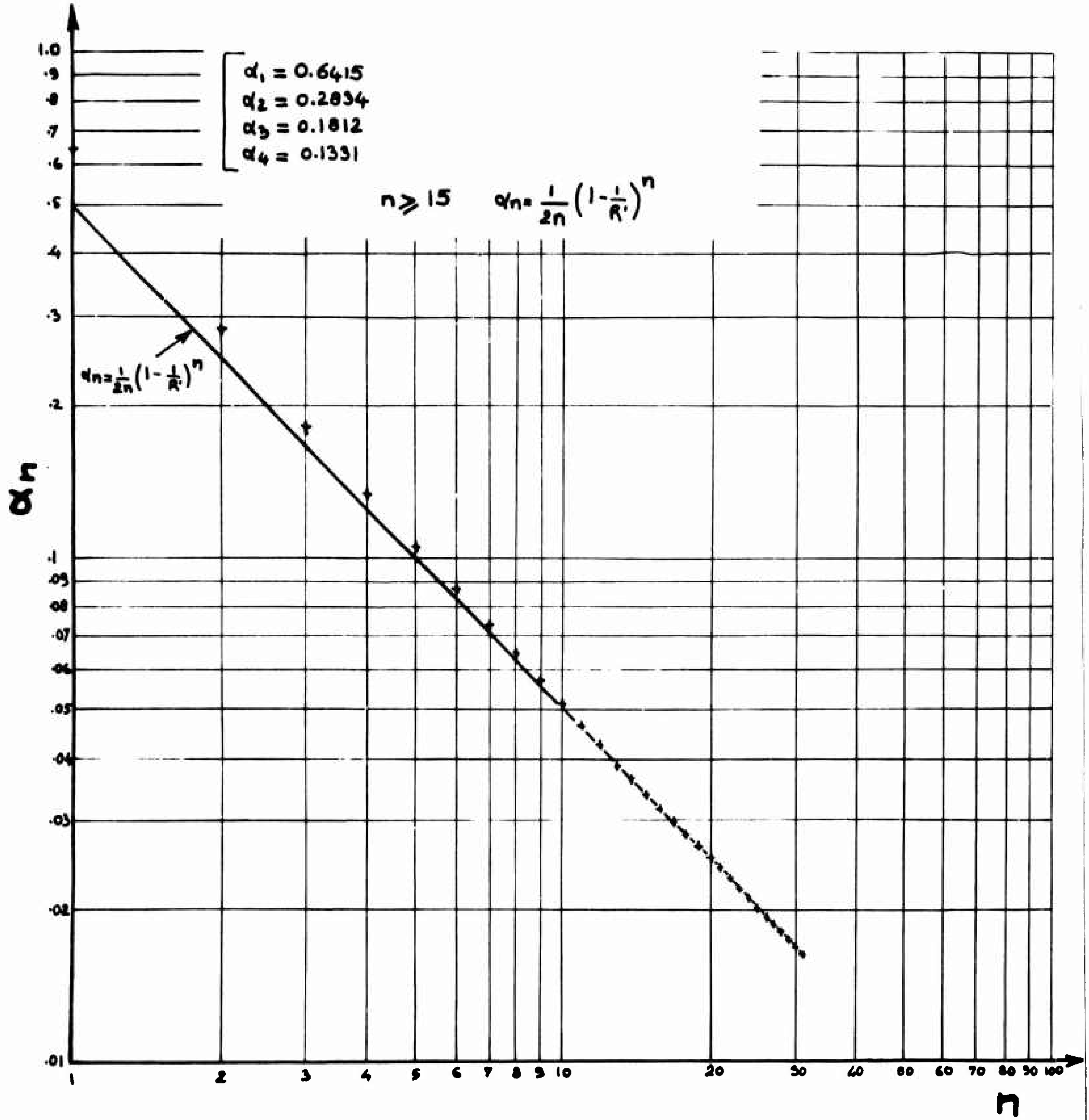
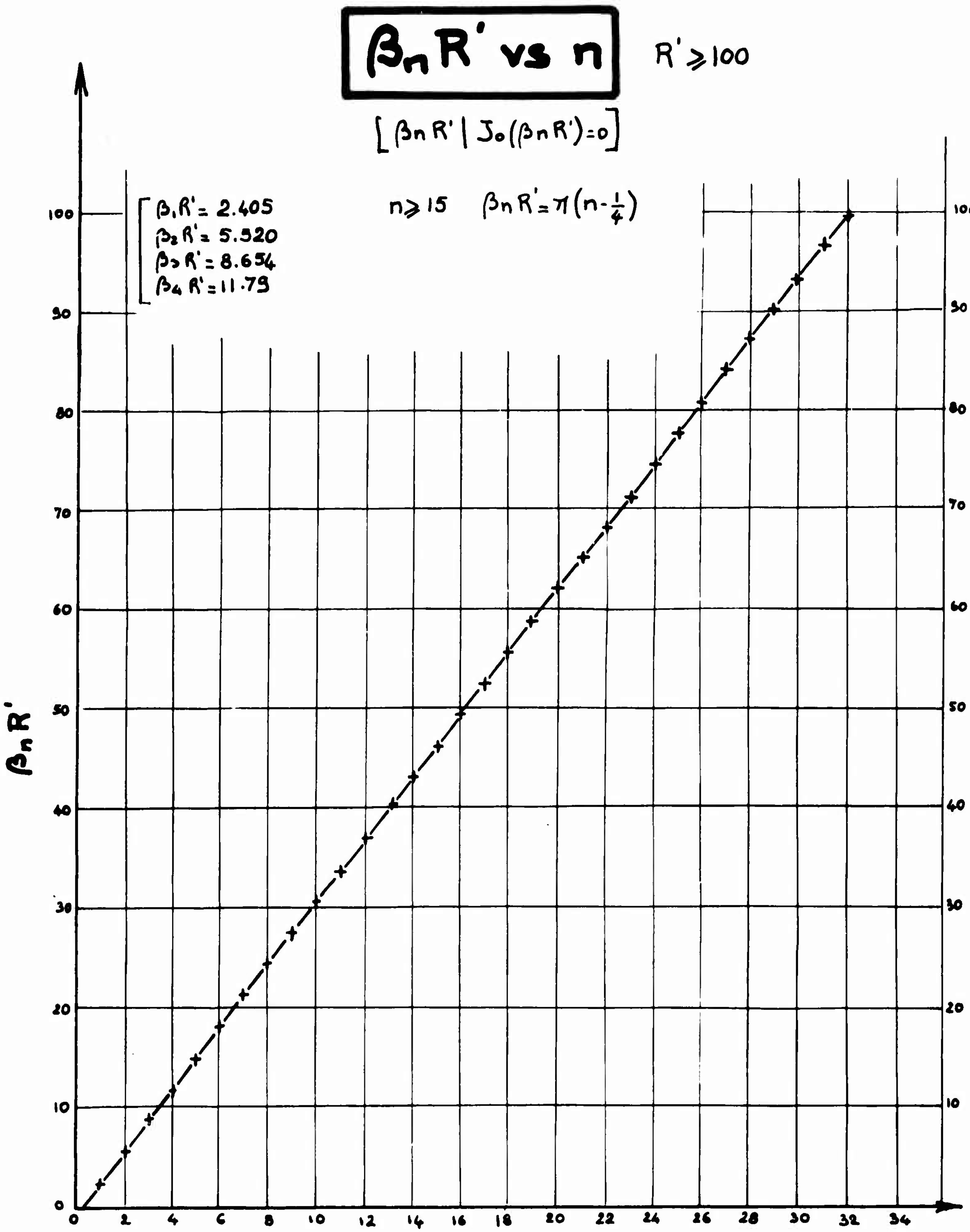


Figure : 19



βnR' vs n
R' ≥ 100
[βnR' | J0(βnR') = 0]
β1R' = 2.405
β2R' = 5.520
β3R' = 8.654
β4R' = 11.79
n ≥ 15 βnR' = π(n - 1/4)
βnR'
n
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
10
20
30
40
50
60
70
80
90
100

figure: 20